Noncommutative realizations: automatic analyticity and more

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The functional calculus

Let $f : (a, b) \to \mathbb{R}$.
Given a self-adjoint matrix A with spectrum in (a, b) diagonalized by a unitary matrix U, that is,

$$A = U^* \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} U$$

we define the expression $f(A)$ via the following formula.

$$f(A) = U^* \begin{pmatrix} f(\lambda_1) & 0 & \dots \\ 0 & f(\lambda_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} U.$$

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Matrix inequalities

Let A and B be self-adjoint matrices:

- We say $A \leq B$ if $B - A$ is positive semi-definite.
- We say $A < B$ if $B - A$ is positive definite.

A positive definite matrix is a self-adjoint matrix with positive eigenvalues, and a positive semi-definite matrix is a self-adjoint matrix with nonnegative eigenvalues.

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Special matrix functions

Let $f : (a, b) \to \mathbb{R}$.

- f is **matrix monotone** if, for any natural number $n \in \mathbb{N}$, and any pair of n by n self-adjoint matrices A and B with spectrum in (a, b),

$$A \leq B \Rightarrow f(A) \leq f(B).$$

- f is **matrix convex** whenever f evaluated on n by n matrices via the matrix functional calculus is a matrix-valued convex function. That is,

$$f\left(\frac{A+B}{2}\right) \leq \frac{f(A)+f(B)}{2}$$

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Classical representations: Löwner-Nevanlinna

Let Π denote the complex upper half plane. A **Pick function** is an analytic map $f : \Pi \to \overline{\Pi}$.

Theorem (Löwner's theorem)

A function $f : (a, b) \to \mathbb{R}$ is matrix monotone if and only if f analytically continues to a Pick function $f : \Pi \cup (a, b) \to \overline{\Pi}$.

Theorem (Nevanlinna's representation)

f is a Pick function if and only if there exist $a \in \mathbb{R}, b \geq 0$, and a positive finite Borel measure μ with $\int \frac{1}{t^2+1} d\mu < \infty$ such that for all $z \in \Pi$

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1}{t-z} - \frac{t}{t^2+1}\, d\mu(t).$$

Moreover, a, b, μ are uniquely determined by f.

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Moreover, a, b, μ are uniquely determined by f.

Classical representations: Kraus

Theorem (Kraus 1937)

Let $f : (-1, 1) \to \mathbb{R}$. f is matrix convex if and only if

$$f(x) = a + bx + \int_{[-1,1]} \frac{x^2}{1 + tx}\, d\mu(t)$$

where $a, b \in \mathbb{R}$ and μ is a finite measure supported on $[-1, 1]$. Note that all such functions analytically continue to the upper half plane.

A circle of ideas

f is matrix monotone on (a, b)

$\Rightarrow$ f can be approximated uniformly with smooth functions that preserve properties of f (mollifiers)

$\Rightarrow$ f is C^1

$\Rightarrow$ f has an infinitely differentiable integral representation against a probability measure ν

$\Rightarrow$ f is real analytic

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Automatic real analyticity

Löwner and Kraus theorems are example of what we might call a **automatic real analyticity theorem** –

$$\begin{aligned} \text{matrix monotone} &\Rightarrow \text{ real analytic representation} \\ &\Rightarrow \text{ analytic continuation to Pick function} \end{aligned}$$

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Questions

Q: Does this heuristic generalize?

Lots of realization/representation analogues to classical theorems in the nc setting. Samples we drew on

- Nevanlinna representations (Agler-T.D.-Young '16, Pascoe-T.D. '17)
- Kraus-type representations (Helton-McCullogh-Vinnikov '06)

Evidence that an automatic analyticity approach might be achievable for noncommutative functions.

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The matrix universe

We define the **matrix universe** to be:

$$\mathcal{M}^d = \bigcup M_n(\mathbb{C})^d,$$

where $M_n(\mathbb{C})^d$ is d-tuples of n by n matrices.
We define the **real matrix universe** to be:

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Noncommutative sets

Free set $D \subseteq \mathcal{M}$:

1. $X, Y \in D \Rightarrow X \oplus Y \in D$
2. X implies $U^* X U \in D$ whenever U is unitary.

Examples: tuples of contractions, tuples of commuting contractions, block 2 by 2 self-adjoint contractions, etc.

We say D is **(open, connected, convex)** whenver each $D \cap M_n(\mathbb{C})$ is (open, connected, convex.)

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Let D be a free set. We define a **(real) noncommutative function** $f : D \to \mathcal{M}$ to satisfy

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Butterfly realization - a Kraus theorem

Theorem (Helton, McCullough, Vinnikov '06)

Let $r : G \subset \mathcal{S}^d \to \mathcal{S}$ denote a noncommutative rational function on a domain G containing 0. If r is matrix convex near 0, then r has a realization of the form

$$r(X) = r_0 + L(X) + \Lambda(X)^*(1 - \Gamma(X))^{-1}\Lambda(X)$$

for a scalar r_0, a real linear function L, Λ affine linear, and $\Gamma(X) = \sum A_i \otimes X_i$ for self-adjoint matrices A_i.

$$f(x) = a + bx + \int_{[-1,1]} \frac{x^2}{1+tx}\, d\mu(t)$$

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A royal proof

What structural features appear in the proofs of automatic analyticity theorems?

- Structured domains
- Local boundedness
- Controlled analyticity on 1 dimensional slices - local domination by derivatives, e.g.
- Amenability to approximation - closure under convolution, e.g.

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A royal proof: dominions

A **dominion** is a set of domains, $\mathcal{G}$, in $\mathbb{R}^d$ satisfying:

- **Translation invariance**: For all $G \in \mathcal{G}$ and $r \in \mathbb{R}^d$, $G + r \in \mathcal{G}$.
- **Scale invariance:** If $t > 0$ and $G \in \mathcal{G}$, $tG \in \mathcal{G}$.
- **Closure under intersection:** For all $G, H \in \mathcal{G}$, $G \cap H \in \mathcal{G}$.
- **Locality:** For any $x \in \mathbb{R}^d$ and $\varepsilon > 0$, $B_{\mathbb{R}}(x, \varepsilon) \in \mathcal{G}$.

The class of convex sets in $\mathbb{R}^d$ is an example of a dominion, as is the class of all open sets in $\mathbb{R}^d$.

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A royal proof: sovereign classes

A **sovereign class** $\mathcal{F}$ is a set of functions on domains contained in a dominion $\mathcal{G}$ satisfying:

- **Functions:** For all $G \in \mathcal{G}$, $\mathcal{F}(G)$ is a set of locally bounded measurable functions.
- **Closure under localization:** If $f \in \mathcal{F}(G)$ and $H \subseteq G$ then $f|_H \in \mathcal{F}(H)$.
- **Closure under convolution:** The set of functions $\mathcal{F}(G)$ is convex and closed under pointwise weak limits.
- **Local boundedness:** Each $f \in \mathcal{F}$ is locally bounded and measurable on finite dimensional affine subspaces on each level.
- **One variable knowledge:** If $a_i \leq b_i$ for each i, then $f_{\overline{ab}}(t) := f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)$ analytically continues to $\mathbb{D}$ as a function of t.
- **Control** There is a map γ taking each pair (x, f) to a non-negative number satisfying
  1. For each $\varepsilon > 0$ there is a universal constant $c(\varepsilon)$ such that $\inf_{X \in B_{\mathbb{R}}(x,\varepsilon)} \gamma(x, f) \leq c(\varepsilon) \|f\|_{B_{\mathbb{R}}(x,\varepsilon)}$.
  2. There is a universal positive valued function e on $\mathbb{R}^+$ satisfying the following. Write $f_{\overline{ab}}(t) = \sum a_n t^n$. Then, $\|a_n\| \leq \gamma(x, f)e(\|b - a\|)$.
  3. If $H \subseteq G$ and $x \in H$ then $\gamma(x, f|_H) \leq \gamma(x, f)$.

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Royal road theorem

Theorem (Pascoe-T.D. 2019)

Every function in a sovereign class is real analytic.

Essentially forced by choosing the correct axioms.

In Pascoe-T.D. 2020 (*Regal Path*), a simplified set of axioms can be used to extract a proof of the commutative Löwner theorem (Agler-McCarthy-Young 2012 and Pascoe 2018) from the noncommutative theorem (Pascoe-T.D. 2016, Palfia 2020, Pascoe 2019)

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A road to realizations

Define the **positive-orthant norm of the n-th derivative at X** by

$$\|D^n f(X)\|_+ = \sup_{\|H\|=1, H>0, m} \|D^n f(X^{\oplus m})[H]\|.$$

If the n-th derivative does not exist in some positive direction, we formally set $\|D^n f(X)\|_+ = \infty$.

Theorem

Matrix monotone functions are a sovereign class.

Controlled by

$$\gamma(X, f) = \|f(X)\| + \|Df(X)\|_+ .$$

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The butterfly realization

Theorem (P.-Tully-Doyle '19)

Let f be a locally bounded matrix convex function defined on some matrix convex set of self adjoints containing 0. There are self-adjoint T_i, vector Q_i, a scalar a_0, and a linear function L such that

$$f(X) = a_0 + L(X) + \left(\sum Q_i X_i\right)^* \left(I - \sum T_i X_i\right)^{-1} \left(\sum Q_i X_i\right).$$

The minimal such realization is essentially unique.

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