1. (a) Show that the complex exponential functions $\{e^{inx}\}_{n=-\infty}^{\infty}$ are an orthonormal system on the interval $(-\pi, \pi)$ with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

   (b) Find a Fourier series expansion for the function $f(x) = e^{2x}$ on the interval $(-\pi, \pi)$.

2. (a) Demonstrate that the set of functions given by

$$\{\cos(2\pi n x)\}_{n=0}^{\infty}$$

   is an orthogonal family on $[-1, 1]$.

   (b) Compute a formula for $||\cos(2\pi n x)||$.

   (c) Find the projection of $f(x) = x^2 + x$ onto the span of $\{\cos(2\pi n x)\}_{n=0}^{\infty}$.

   (d) Is this series equal to $f(x)$ on $(-1, 1)$?

3. Consider the equation

$$u_t = u_{xx} - (1 - t)u$$

   with boundary conditions $u_x(t, 0) = u_x(t, 1) = 0$.

   (a) Find the general solution to the problem.

   (b) Find the solution in the case that the initial condition is

$$u(0, x) = f(x) = \cos(2\pi x) - 3\cos(3\pi x).$$

4. (Much more interesting) Consider the diffusion-convection problem

$$\begin{aligned} DE &: u_t = u_{xx} + 2u_x \\ BC &: u(t, 0) = u(t, \pi) = 0 \\ IC &: u(0, x) = e^{-x}(3\sin x - 2\sin 4x). \end{aligned}$$

   (a) Find the general separable solution. (Hint: you might consider the change of variables $X(x) = e^{-x}Y(x)$ in the spatial equation).

   (b) Solve the initial value problem.