So far, we've been pretty cagey about what it means to "converge" as a Fourier series.

if $\tilde{f}(x)$ is the 2π-periodic extension of f, we know that

$$\tilde{f}(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

if $\tilde{f}$ is piecewise continuous and $\bar{f}$ is continuous at x.

but this hasn't really answered the question of "which f possess a convergent Fourier series"?

we're going to describe a vector space in which <u>every</u> function has a Fourier series representation.

A **norm** is a magnitude measuring function.

$$\begin{cases} \|x\| = 0 \Rightarrow x = 0 \\ \|x\| \geq 0 \text{ for all } x \in V. \end{cases}$$

$\|cx\| = |c|\,\|x\|$ for all scalars c, vectors x.

$\|x+y\| \leq \|x\| + \|y\|$.

Given a normed vector space V, we can measure the distance between vectors

$$d(x,y) = \|x-y\|$$

An inner product $\langle \cdot , \cdot \rangle$ on V defines a norm by

$$\|x\|^2 = \langle x, x \rangle.$$

Def: A function f is square-integrable on $[-\pi, \pi]$ if

$$\|f\|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$$

(intrinsically complex)

The Hilbert space L^2 is the vector space of all square integrable functions on $[-\pi, \pi]$.

$$L^2[-\pi,\pi] = L^2 = \{ f : [-\pi,\pi] \to \mathbb{C} \mid \|f\|_2 < \infty \}.$$

What sort of functions are in L^2?

- piecewise continuous functions.
- some function w/ singularities

$$f(x) = \frac{1}{x^{1/3}}$$

- additions like the Dirichlet function

$$d(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

Hilbert spaces are <u>complete</u> in the sense that sequences in L^2 that look convergent have limits in L^2.

This requires a notion of integrability that goes well beyond the Riemann integral.

For example, $\frac{1}{2\pi}\int_{-\pi}^{\pi} d(x)\,dx$ is not defined as a Riemann integral,

but $\frac{1}{2\pi}\int_{-\pi}^{\pi} d(x)\,dx = 0$ as a Lebesgue integral.

Why do we need L^2? The continuous functions are not complete.

Think about the sequence

$$f_n(x) = \left(\frac{x}{\pi}\right)^{2n} \quad \text{on } [-\pi,\pi]$$

each function is continuous, and the limit exists pointwise everywhere:

$$f(x) = \begin{cases} 1 & \text{if } x = \pm\pi \\ 0 & \text{otherwise.} \end{cases}$$

but f isn't continuous.

Def: Let V be a normed vector space. A sequence $\{s_n\}$ converges in norm to $f \in V$ if $\|s_n - f\| \to 0$ as $n \to \infty$.

Let V be an inner product space.

Thm: Given an orthonormal set of vectors $\varphi_1, \ldots, \varphi_n \in V$, the closest vector in $\text{span}\{\varphi_1 \ldots \varphi_n\} \subset V$ to $f \in V$ is

$$\hat{f} = \text{proj}_{\text{span}\{\varphi_i\}} f = \sum_{i=1}^{n} \langle f, \varphi_i \rangle \varphi_i$$

in the sense that $\| f - \sum_{i=1}^{n} c_i \varphi_i \|$ is minimized by $\hat{f}$.

Def: An orthonormal system $\varphi_1 \ldots \in V$ is called complete if for every $f \in V$,

$$s_n = \sum_{k=1}^{n} \langle f, \varphi_k \rangle \varphi_k$$

has

$$\| f - s_n \| \to 0 \text{ as } n \to \infty.$$

Theorem (way beyond the scope of this course)

The trigonometric functions $1, \cos kx, \sin kx$ are a complete orthogonal set in $L^2[-\pi,\pi]$. That is, if $f \in L^2$, then $\left\| f - \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \right) \right\| \to 0$ as $n \to \infty$.

<u>Caveat:</u> suppose that $f(x) = 0$ on $(-\pi, \pi)$

and $g(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$ on $(-\pi, \pi)$.

$\| f(x) - g(x) \| = 0.$

So by the definition of the norm, f should be g.

So functions in L^2 are actually equivalence classes.

The problems we'll be solving will give L^2 functions represented by Fourier series as solutions.

L^2 is the most important vector space of solutions to PDEs.