Our objective is to figure out a way to write $f(x)$ as a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_k \cos kx + b_k \sin kx.$$

First, we should figure out how to find a_k, b_k.

Many of you have learned dumb formulas for this that don't generalize.

Our goal is to make this a linear algebra problem. In finite dimensions, given an n dimensional vector space V with an inner product $\vec{v} \cdot \vec{w}$, we say $\vec{v} \perp \vec{w}$ if $\vec{v} \cdot \vec{w} = 0$. An <u>orthonormal basis</u> for V is a collection of vectors $\vec{u}_1, \dots, \vec{u}_n$ so that

$$\vec{u}_i \cdot \vec{u}_j = 0 \qquad i \neq j$$
$$= 1 \qquad i = j$$

and each $v \in V$ can be written

$$\vec{v} = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n.$$

Importantly, the coefficients c_i are extremely easy to compute:

$$c_i = \vec{v} \cdot \vec{u}_i .$$

So given o.n. basis $\{\vec{u}_i\}$, every $\vec{v} \in V$ can be expressed

$$\vec{v} = \sum_{k=1}^{n} (\vec{v} \cdot \vec{u}_k)\, \vec{u}_k .$$

So we'd like to construct a vector space of functions with $\cos kx, \sin kx$ as an orthonormal basis.

Def: Given functions f, g on $[-\pi, \pi]$, the L^2-inner product $\langle f, g \rangle$ is given by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx .$$

The L^2-norm of f is

$$\| f \| = \sqrt{\langle f, f \rangle} \quad \text{or} \quad \| f \|^2 = \langle f, f \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx .$$

You might show that this is indeed an inner product.

Lemma: $\langle \cos kx, \cos lx \rangle = 0$ $\quad k \neq l$

$\langle \sin kx, \sin lx \rangle = 0$ $\quad k \neq l$

$\langle \cos kx, \sin lx \rangle = 0$ $\quad$ for all k, l.

$\| 1 \| = \sqrt{2}$

$\| \cos kx \| = 1$ $\quad$ for all k

$\| \sin kx \| = 1$ $\quad$ for all k.

That is, $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$

is an orthogonal system of functions.

How can we use this to compute coefficients?

Suppose $f(x) = \frac{a_0}{2} + \sum_k a_k \cos kx + b_k \sin kx$.

then $\langle f, \cos lx \rangle$

$$= \langle \frac{a_0}{2} + \sum_k a_k \cos kx + b_k \sin kx, \cos lx \rangle$$

$$= \cancel{\langle \frac{a_0}{2}, \cos lx \rangle} + \sum_k a_k \underbrace{\langle \cos kx, \cos lx \rangle}_{\substack{0 \\ \text{unless } k = l}} + b_k \cancel{\langle \sin kx, \cos lx \rangle}$$

0 $\qquad$ 0

$$= a_l \langle \cos lx, \cos lx \rangle$$

$$= \underline{a_l}.$$

so $a_\ell = \langle f, \cos \ell x \rangle$.

likewise, $b_\ell = \langle f, \sin \ell x \rangle$

$$a_0 = \langle f, 1 \rangle \quad \text{(hence the 2).}$$

so $\boxed{\text{if}}$ $f(x) = \frac{a_0}{2} + \sum_k a_k \cos kx + b_k \sin kx$

then $f(x) = \frac{1}{2}\langle f, 1 \rangle + \sum_k \langle f, \cos kx \rangle \cos kx + \langle f, \sin kx \rangle \sin kx.$

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lets compute one, leaving aside questions of convergence for now.

Ex: let $f(x) = x$ on $[-\pi, \pi]$.

$$\langle x, 1 \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi} x^2 \Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(\pi^2 - (\pi^2)\right) = 0.$$

$$\langle x, \cos kx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos kx\,dx$$

x	$\cos kx$
1	$\frac{1}{k}\sin kx$
0	$-\frac{1}{k^2}\cos kx$

$$\frac{1}{k} x \sin kx + \frac{1}{k^2}\cos kx \Big|_{-\pi}^{\pi}\Big)$$

$$= \frac{1}{\pi}\left(\left(0 + (-1)^k\right) - \left(0 + (-1)^k\right)\right) = 0.$$

$$\langle x, \sin kx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx\,dx$$

$$= \frac{1}{\pi}\left(-\frac{x}{k}\cos kx + \frac{1}{k^2}\sin kx \right) \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\left(-\frac{\pi}{k}(-1)^k + 0 \right) - \left(\frac{\pi}{k}(-1)^k \right) \right)$$

$$= -\frac{2}{k}(-1)^k$$

$$= \frac{2}{k}(-1)^{k+1}$$

So $$x \sim \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k} \sin kx.$$

$$= 2\sin x - \frac{2\sin 2x}{2} + \frac{2\sin 3x}{3} - \frac{2\sin 4x}{4} + \dots$$

convergence is tricky. ~ means "has the Fourier representation"

So what are we doing here? basically projecting f onto the span of $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$

A Fourier series is not a nice object like a Taylor series.

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and it converges, automatically

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$ and so on.

what about our example?

$$x \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} 2\sin(nx)$$

$$\frac{d}{dx}$$

$$1 \underset{c}{\overset{?}{\sim}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\cancel{n}} \cdot 2\cos(nx)\cancel{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} 2\cos(nx)$$. uh oh...

nope