the wave equation was

$$u_{tt} = c^2 u_{xx}.$$

on the line, we could use d'Alembert's method. (Simple)

Let us now consider the <u>heat</u> <u>equation</u>

$$u_t = u_{xx}.$$

This doesn't have an elementary formula for its general solution.

instead, we'll write solutions as infinite series of families of simple, explicit sol$^{\underline{s}}$

this chapter will develop the theory of these series, <u>FOURIER</u> <u>SERIES</u>.

The *evolutionary form* of the heat equation is $u_t = L[u]$ where $L = \frac{\partial^2}{\partial x^2}$.

recall $L[cu + dv] = cL[u] + dL[v]$.

also, since L only depends on x,

$$L[c(t)u] = c(t)L[u].$$

Other operators are possible.

$$L[u] = -c(x)\frac{\partial u}{\partial x} \qquad \text{transport}$$

$$L[u] = \frac{\partial^2 u}{\partial x^2} - \gamma u \qquad \text{damped heat eq}^{n}.$$

Fourier's Idea:

Consider

$$\frac{du}{dt} = \lambda u$$

This has general solution $u = ce^{\lambda t}$

in linear algebraic notation,

$$D[u] = \lambda u$$

eigen vector

eigen value

First extension: given a first order homogeneous system of ODEs,

$$\frac{du_1}{dt} = 2u_1 + u_2$$

$$\frac{du_2}{dt} = 3u_1 + 2u_2$$

$$\begin{bmatrix} du_1/dt \\ du_2/dt \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

$$\frac{d\vec{u}}{dt} = A\vec{u}.$$

guess $\vec{u} = e^{\lambda t}\vec{v}$ for some constant $\vec{v}$

just like $u = ce^{\lambda t}$.

$$\frac{d\vec{u}}{dt} = \lambda e^{\lambda t} \vec{v}$$

$$\lambda e^{\lambda t} \vec{v} = A(e^{\lambda t}\vec{v})$$

$$\lambda \vec{v} = A\vec{v}.$$

so $\vec{u} = e^{\lambda t}\vec{v}$ is a solution if

$\lambda, \vec{v}$ are an eigenpair of A.

$\vec{u}(t) = e^{\lambda t}\vec{v}$ is called an <u>eigensolution</u>

of $\frac{d\vec{u}}{dt} = A\vec{u}$.

If A has an eigenbasis $\vec{v}_1, \dots \vec{v}_n$

then the general solution to $\frac{d\vec{u}}{dt} = A\vec{u}$

is $\vec{u} = \sum_{i=1}^{n} c_i e^{\lambda_i t} \vec{v}_i.$

the eigensolutions form a basis for the general solution set.

$$\frac{du}{dt} = \lambda u$$

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

$$¿\ \frac{\partial u}{\partial t} = L[u]\ ?$$

let's try $u(t,x) = e^{\lambda t} v(x)$

$$\frac{\partial u}{\partial t} = \lambda e^{\lambda t} v(x)$$

$$L[u] = L\left[e^{\lambda t} v(x)\right] = e^{\lambda t} L[v(x)]$$

$$\frac{\partial u}{\partial t} = L[u]$$

$$\lambda \cancel{e^{\lambda t}} v(x) = \cancel{e^{\lambda t}} L[v(x)]$$

So v must satisfy the eigenequation

$$L[v] = \lambda v.$$ This is a function only of x.

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let us consider $L[v] = \frac{\partial^2 v}{\partial x^2} = v''$

$v'' = \lambda v.$ suppose $\lambda = \omega^2 > 0$

$$v'' - \omega^2 v = 0$$

$$m^2 - \omega^2 = (m+\omega)(m-\omega) = 0$$

$$m = \pm \omega.$$

$$v = e^{\omega x}, e^{-\omega x}$$

eign solutions: $u(t,x) = e^{\omega^2 t} e^{\omega x} \quad e^{\omega^2 t} e^{-\omega t}.$

$\lambda = -\omega^2 < 0.$

$$v'' + \omega^2 v = 0$$

$m^2 + \omega^2 = 0 \qquad m = \pm \omega i$

$v = \cos(\omega x), \sin(\omega x)$ (or $e^{i\omega x}$ $e^{-i\omega x}$)

$u(t,x) = e^{-\omega^2 t} \cos \omega x \quad e^{-\omega t} \sin \omega x.$

$\underline{\lambda = 0}$

$v'' = 0$

$v = 1, x$

$u(t,x) = 1, x.$

Absent any additional conditions,

<u>every</u> real λ is an eigenvalue!

Any finite linear combination of sol^{s} is a sol, so e.g.

$$u(t,x) = c_1 e^{-t} \cos(x) + c_2 e^{-4t} \sin(2x) + c_3 + c_4 x$$

is a solution.

A note to consider:

$\lambda < 0$ gives solutions like $e^{-\lambda t} \cos \omega x$.

physically reasonable.

$\lambda > 0$ gives solutions like $e^{\lambda t} e^{\omega x} \to \infty$ as $t \to \infty$.

<u>physically bad!</u>

To get more information, we need to apply boundary conditions. Let us consider Fourier's problem.

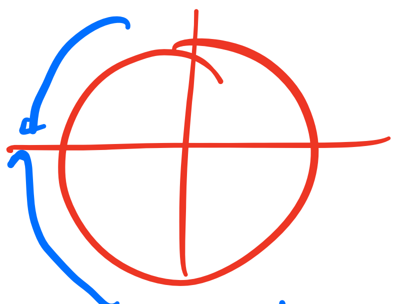
Consider the heat equation

$$u_t = u_{xx} \quad \text{on} \quad -\pi \le x \le \pi \quad \text{subject to}$$

$$\begin{cases} u(t,-\pi) = u(t,\pi) \\ u_x(t,-\pi) = u_x(t,\pi) \end{cases} \quad \text{for } t > 0$$

These are called periodic boundary conditions.

$u(t,x)$ is temp on an insulated metal ring. so



which of our solution can meet this?

if $u(t,x) = e^{\lambda t} v(x)$

the boundary conditions become

$$e^{\lambda t} v(-\pi) = e^{\lambda t} v(\pi) \qquad t>0$$

so $\boxed{v(-\pi) = v(\pi)}$

and $e^{\lambda t} v'(-\pi) = e^{\lambda t} v'(\pi) \qquad t>0$ so

$$\boxed{v'(-\pi) = v'(\pi)}$$

for the ODE $v'' = \lambda v$.

we seek nontrivial solution.

If $\lambda = \omega^2 > 0$

the general solution is

$$v(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

applying bdy conditions give

$$\underbrace{c_1 e^{-\omega\pi} + c_2 e^{\omega\pi}}_{v(-\pi)} = \underbrace{c_1 e^{\omega\pi} + c_2 e^{-\omega\pi}}_{v(\pi)}$$

$(c_1 - c_2) e^{-\omega\pi} + (c_2 - c_1) e^{\omega\pi} = 0$

$\Rightarrow c_1 - c_2 = 0$

$c_1 = c_2$.

so $c_1 = c_2$ since $\omega \neq 0$.

also $v'(x) = c_1 \omega e^{\omega x} - c_2 \omega e^{-\omega x}$

so $c_1 w e^{-\omega\pi} - c_2 w e^{\omega\pi} = c_1 w e^{\omega\pi} - c_2 w e^{-\omega\pi}$

$\Rightarrow \quad c_1 = -c_2.$

$c_1 = c_2$ and $c_1 = -c_2$

require that $c_1 = c_2 = 0$.

so no nontrivial solutions.

if $\lambda = 0$, $v = a + bx$

$v' = b$

$a - b\pi = a + b\pi$ and $b = b$

$\Rightarrow \quad b = 0$

so $v(x) = a$ works. $\lambda = 0$ is an eigenvalue.

take $v_0(x) \equiv 1$.

$$u(t,x) = e^{0t} v_0(x) = 1$$

Finally, $\lambda = -\omega^2 < 0$.

$v(x) = a \cos \omega x + b \sin \omega x$

$v'(x) = -a\omega \sin(\omega x) + b\omega\cos(\omega x)$

$a\cos(-\omega\pi) + b\sin(-\omega\pi) = a\cos(\omega\pi)$
$+ b\sin(\omega\pi)$

$\boxed{\Rightarrow\ 2b\sin(\omega\pi) = 0}$

$-a\omega\sin(-\omega\pi) + b\omega\cos(-\omega\pi)$
$= -a\omega\sin(\omega\pi) + b\omega\cos(\omega\pi)$

$2a\sin(\omega\pi) = 0$

if $\sin(\omega\pi) \neq 0$ $a, b = 0$ trivial sol.

so assume $\sin(\omega\pi) = 0$

$\Rightarrow \omega\pi = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

(but $\omega > 0$) so

$\omega\pi = \pi, 2\pi, 3\pi, 4\pi \ldots$

so $\omega = 1, 2, 3, 4, 5 \ldots$

Hence $v(x) = a\cos(kx) + b\sin(kx)$ are
eigen solutions

and $\lambda_k = -k^2$ is an eigenvalue, each of which has a 2d eigenspace.

$$u_k(x) = e^{-k^2 t}\cos(kx) \quad \tilde{u}_k(t,x) = e^{-k^2 t}\sin(kx)$$

are eigenfunction solutions to the BCD.

$k = 0, 1, 2, \ldots$ (0 to include 1).