The wave equation on the real line.

A model for small displacements on a 1D medium (like a string) is

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)$$

mass density

acceleration

tension

slope of displacement

This is essentially Newton's second law and elbow grease.

To make this tractable, let's assume small disturbances on a medium of uniform density ρ and stiffness k.

Then this simplifies to the $\boxed{\text{1D wave equation}}$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{where} \quad c = \sqrt{k/\rho}$$

Since we have two time derivatives, we need two initial conditions to solve this, say

$$u(0,x) = f(x) \qquad \frac{\partial u}{\partial t}(0,x) = g(x).$$

This is an <u>initial value problem</u>.

PDE: $u_{tt} = c^2 u_{xx}$

IC: $u(0,x) = f(x)$

$u_t(0,x) = g(x)$

we seek a C^2 solution $u(t,x)$ meeting the initial conditions.

Let $\Box = \partial_t^2 - c^2\partial_x^2$ be the <u>wave</u> operator.

then the wave equation can be written

$\Box u = 0.$

we can factor the operator

$$\Box = \partial_t^2 - c^2\partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$$

implicitly, this equality depends on $\partial_t\partial_x u = \partial_x\partial_t u$, Clairaut's Thm, so we <u>must</u> have C^2 solutions.

$\Box u = (\partial_t - c\partial_x)(\partial_t + c\partial_x) u = 0$

has a solution u if $\partial_t + c\partial_x$ annihilates u.

If $(\partial_t + c\partial_x) u = 0$,

then $(\partial_t - c\partial_x)(\partial_t + c\partial_x) u$

$= (\partial_t - c\partial_x) 0$

$= 0$

of course, $(\partial_t + c\partial_x) u = 0$ is a transport equation

$$u_t + c u_x = 0$$

which is solved by

$u(t,x) = p(x-ct)$,

a rightward traveling wave w/ speed c for an arbitrary C^2 function p.

so $u(t,x) = p(x-ct)$ is a family of solutions to $u_{tt} = c^2 u_{xx}$.

of course we could have factored the operator in the other order:

$$\Box = (\partial_t + c\partial_x)(\partial_t - c\partial_x)$$

and so solutions to

$$(\partial_t - c\partial_x) u = 0$$

are also solutions to

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x) u = 0.$$

$u_t - c u_x = 0$ is a transport eq^n

with solutions

$$u = q(x+ct) \qquad q \text{ arb, } C^2.$$

Leftward traveling wave w/ speed c.

by linearity,

$$u(t,x) = p(x-ct) + q(x+ct)$$

is also a solution to

$\Box u = 0.$

d'Alembert showed that <u>all</u>

solutions to $\Box u = 0$ have this form of the sum of a left- and right-traveling wave.

<u>Theorem:</u> Every solution to

$u_{tt} = c^2 u_{xx}$ can be written as a superposition of left and right traveling waves:

$$u(t,x) = p(\xi) + q(\eta)$$
$$= p(x-ct) + q(x+ct)$$

where p, q are arb. C^2 functions depending on the characteristic variables

$$\xi, \eta.$$

Pf: Proceed by change of variables.

Let $u(t,x) = v(x-ct, x+ct) = v(\xi, \eta)$

then $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta} \cdot \frac{\partial \eta}{\partial t}$

$$u_t = v_\xi(-c) + v_\eta(c)$$

$$= c(-v_\xi + v_\eta)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$u_x = v_\xi + v_\eta$$

$$u_{tt} = c^2(v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta})$$

$$u_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}$$

then $\Box u = u_{tt} - c^2 u_{xx} = -4c^2 v_{\xi\eta}$

so $0 = \Box u$ iff *

$$0 = v_{\xi\eta}$$

Now write $\frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial \eta}\right) = 0.$

$$= \frac{\partial}{\partial \xi}(w) = 0 \quad \text{where} \quad w = \frac{\partial v}{\partial \eta}.$$

this implies that

$$w = r(\eta), \quad r \text{ arbitrary function of } \eta.$$

sub back

$$w = \frac{\partial v}{\partial \eta}$$

$$r(\eta) = \frac{\partial v}{\partial \eta}$$

so integrating,

$$v = \underbrace{\int r(\eta)\, d\eta}_{} + p(\xi)$$

$$q(\eta) + p(\xi), \quad p, q \text{ arbitrary.}$$

$$v(\xi, \eta) = p(\xi) + q(\eta)$$

means

$$u(t, x) = p(x - ct) + q(x + ct)$$

An illustrative example:

suppose $u_{tt} - 5u_{tx} + 4u_{xx} = 0$

$u(0,x) = \sin x$ position

$u_t(0,x) = \sin x$ velocity

since $(\partial_t^2 - 5\partial_{tx} + 4\partial_x^2)u = 0$

$$(\partial_t - 4\partial_x)(\partial_t - \partial_x)u = 0,$$

the method from the solution of the wave equation gives

$$u = f(x+4t) + g(x+t) \qquad f, g \ C^2$$

I $\quad u(0,x) = f(x) + g(x) = \sin x$

II $\quad u_t(t,x) = 4f'(x+4t) + g'(x+t)$

I' $\quad f'(x) + g'(x) = \cos x$

II' $\quad 4f'(x) + g'(x) = \sin x$

$$3f' = \sin x - \cos x$$

$$f' = \tfrac{1}{3}\sin x - \tfrac{1}{3}\cos x$$

$$f(x) = -\tfrac{1}{3}\cos x - \tfrac{1}{3}\sin x$$

also, $f(x) + g(x) = \sin x$

$$\left(-\frac{1}{3}\cos x - \frac{1}{3}\sin x\right) + g(x) = \sin x$$

$$g(x) = \frac{4}{3}\sin x + \frac{1}{3}\cos(x)$$

so $u(t,x) = f(x+4t) + g(x+t)$

$$= \frac{-1}{3}\cos(x+4t) - \frac{1}{3}\sin(x+4t)$$

$$+ \frac{4}{3}\sin(x+t) + \frac{1}{3}\cos(x+t)$$

Solving the IVP:

Now lets solve

DE: $u_{tt} = c^2 u_{xx}$

IC: $u(0,x) = f(x)$

$u_t(0,x) = p(x)$

$u(t,x) = p(x-ct) + q(x+ct)$

$u(0,x) = \boxed{p(x) + q(x) = f(x)}$

$u_t(t,x) = -c\,p'(x-ct) + c\,q'(x+ct)$

$u_t(0,x) = \boxed{-c\,p'(x) + c\,q'(x) = g(x)}$

since $p(x) + q(x) = f(x)$

I: $p'(x) + q'(x) = f'(x)$

II: $-cp'(x) + cq'(x) = g(x)$

$\mathrm{I} + (-\frac{1}{c})\mathrm{II}$:

$$2p'(x) = f'(x) - \frac{1}{c}g(x)$$

so $p(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(z)\,dz + a$

then $q(x) = f(x) - p(x)$

$$= f(x) - \left(\frac{1}{2} f + \frac{1}{2c} \int_0^x g(z)\,dz + a\right)$$

$$q(x) = f(x) - \frac{1}{2c} \int_0^x g(z)\,dz + a$$

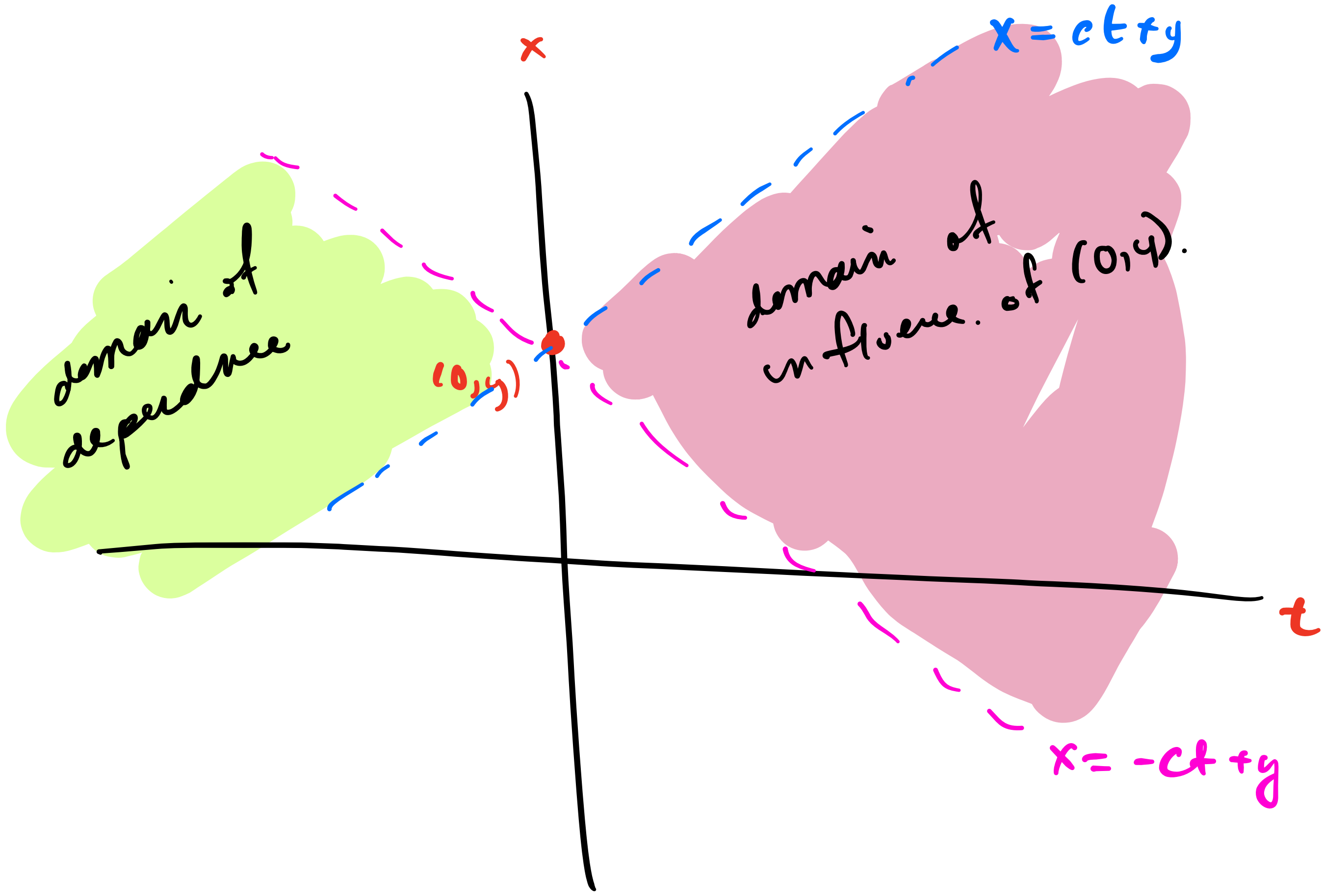
$$u(t,x) = p(\xi) + q(\eta)$$

$$= \frac{f(\xi) + f(\eta)}{2} + \frac{1}{2c} \int_\eta^\xi g(z)\,dz$$

$$= \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\,dz$$

this is called d'Alembert's solution

the characteristics of the wave equation are $\xi = x - ct$ $\quad \eta = x + ct$



Thm 2.18 the solution to the forced IVP

$$u_{tt} = c^2 u_{xx} + F(x,t), \quad u(0,x) = f(x), \quad u_t(0,x) = g(x)$$

$-\infty < x < \infty$

$t > 0$ is

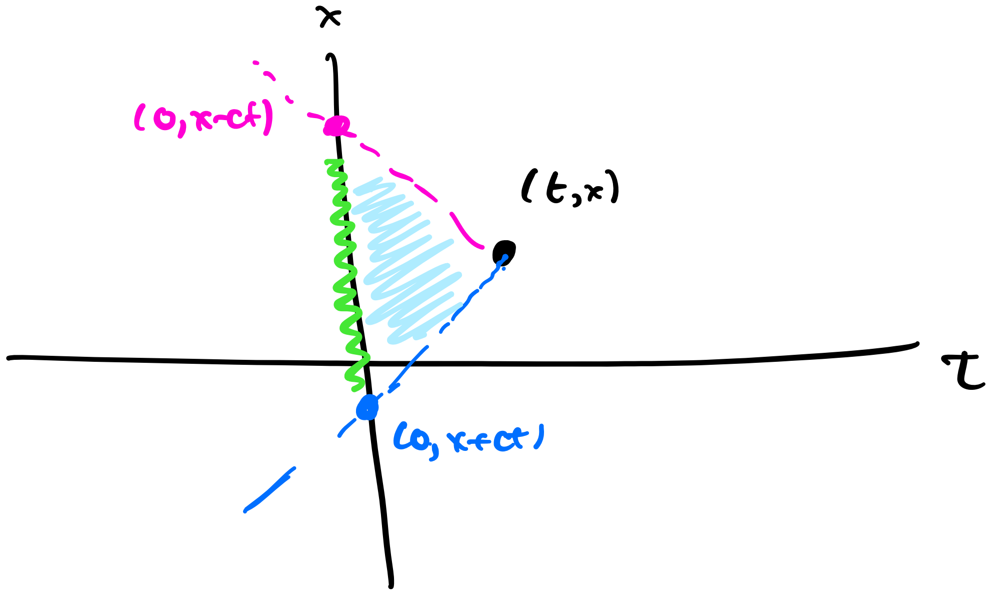
influence of displacement

influence of velocity

$$u(t,x) = \boxed{\frac{f(x-ct) + f(x+ct)}{2}} + \boxed{\frac{1}{2c}\int_{x-ct}^{x+ct} g(x)\,dx}$$

$$+ \boxed{\frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s,y)\,dy\,ds}$$

influence of forcing


x
(0, x−ct)
(t, x)
t
(0, x+ct)