Non-uniform transport

Now lets consider a variable wave speed $c(x)$.

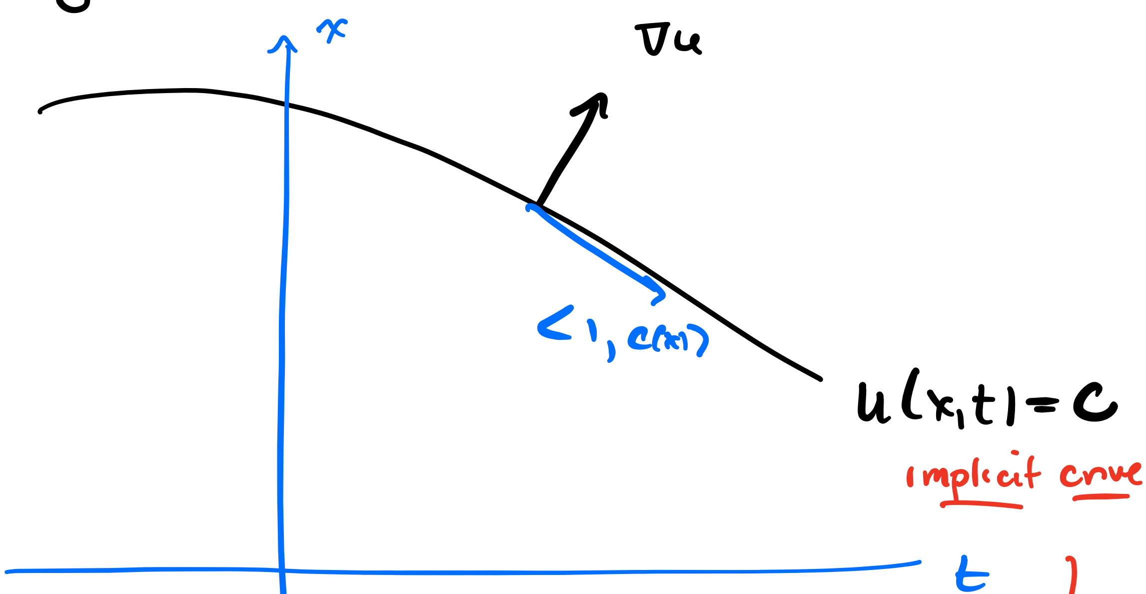
$$u_t + c(x)u_x = 0.$$

as before,

view this as

$$\langle 1, c(x)\rangle \cdot \nabla u = 0$$

That is, the level curves of $u(x,t)$ are given by



$$\frac{dx}{dt} = \frac{c(x)}{1}$$

x is implicitly a function of t

$$\frac{dx}{dt} = c(x).$$

$$\frac{dx}{c(x)} = dt$$

$$\underbrace{\int \frac{dx}{c(x)}}_{\beta(x)} = t + k$$

$\beta(x) = t + k$ implicitly defining a solution

Note:

if $c(x_*) = 0$, then $\frac{dx}{dt} = 0$ and $x = x_*$ is a stationary curve.

$u(t,x)$ is constant along curves $\beta(x) = t + k$.

so let $\underline{\xi = \beta(x) - t}$ be the characteristic variable.

then $u(t,x) = v(\xi)$ for an arbitrary C^1 f: v since all that matters is which curve we're on.

Ex: $\begin{cases} u_t + x u_x = 0 \\ u(0,x) = \dfrac{1}{1+x^2} \end{cases}$

$\langle 1, x\rangle \cdot \nabla u = 0$

characteristic curve:

$$\frac{dx}{dt} = x$$

$$\frac{dx}{x} = dt$$

$$\ln(x) = t + C$$

$$e^{\ln(x)} = e^{t+C}$$

$$x = \xi e^{t}$$ char curves

$$\boxed{x e^{-t} = \xi}$$

Solution is

$$u(t,x) = v(\xi)$$
$$= v(x e^{-t})$$

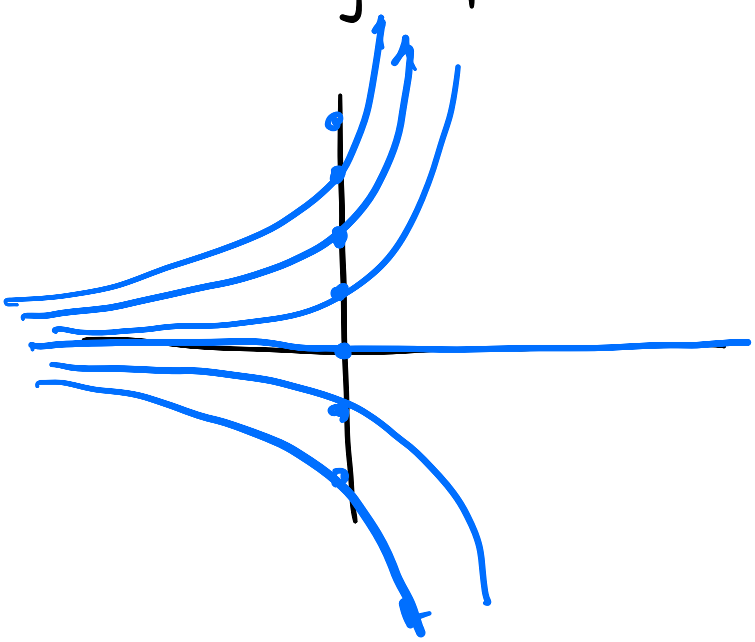
$$u(0,x) = v(x)$$
$$\| $$
$$\frac{1}{1+x^2}$$

$$v(\xi) = \frac{1}{1+\xi^2}$$

$$v(xe^{-t}) = \frac{1}{1+(xe^{-t})^2}$$

$$u(t,x) = \frac{1}{1+(xe^{-t})^2}$$

stationary point at $x_* = 0$ (since $\frac{dx}{dt} = 0$ at $x_0 = 0$)



Ex: $\begin{cases} u_t + \dfrac{1}{1+x^2}\, u_x = 0 \\ u(0,x) = \dfrac{1}{1+(x+3)^2} \end{cases}$

$$\left\langle 1, \frac{1}{1+x^2} \right\rangle \cdot \nabla u = 0$$

char curves

$$\frac{dx}{dt} = \frac{1}{1+x^2}$$

$$(1+x^2)\,dx = dt$$

$$x + \frac{1}{3}x^3 = t + k$$

$$\underbrace{x + \frac{1}{3}x^3 - t = k}_{\text{characteristic curves – implicit}}$$

$$u(t,x) = v(\xi)$$

$$= \frac{1}{1+(\xi+3)^2}$$

$$= \frac{1}{1+\left(x+\frac{1}{3}x^3 - t + 3\right)^2}$$