Stationary waves

Let us begin with a very simple example:

$$u_t = 0.$$

appealing to the FTC,

$$\int_0^t \frac{\partial u}{\partial t}(s,x)\,ds = \int_0^t 0\,ds$$

$$u(t,x) - u(0,x) \qquad 0 \text{ for all } t$$

so $u(t,x) - u(0,x) = 0$

or $u(t,x) = \underbrace{u(0,x)}_{\text{initial condition at } t=0}$

so the solution is

$u(t,x) = f(x)$ where $f(x) = u(0,x)$.

(and f is C^1 so that u is a

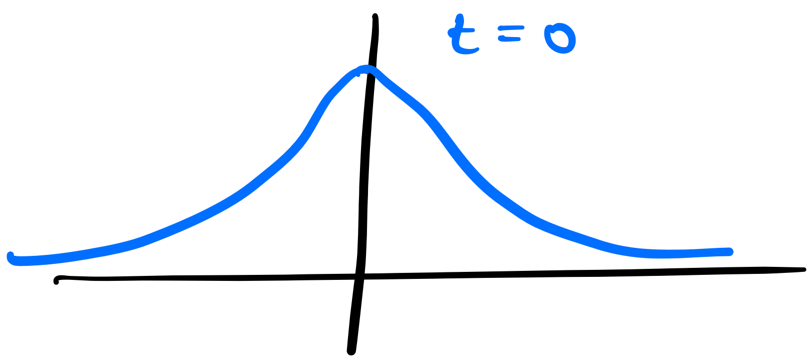
classical solution)

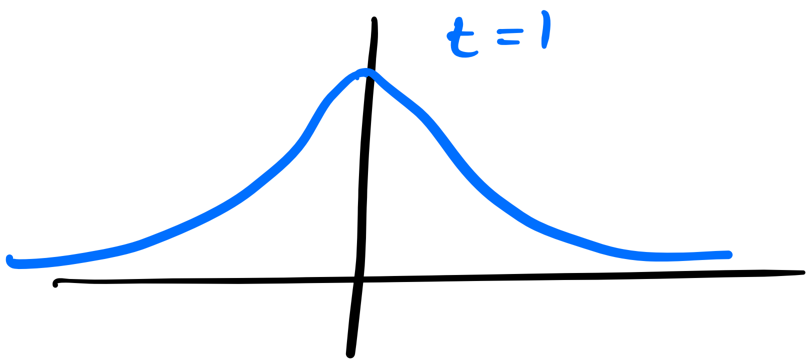
$u(t,x)$ does not change wrt time, so $u(t,x)$ is called a <u>stationary</u> wave.

<u>ex</u>: solve $u_t = 0$ $\quad u(0,x) = e^{-x^2}$

$$u(t,x) = f(x)$$
$$e^{-x^2} = u(0,x) = f(x)$$
$$\text{so } u(t,x) = e^{-x^2}.$$

(boring!)





A more interesting example:

Consider $u_t + 3u_x = 0$.

Reduce to an equation we can solve.
how?

One approach:

$$0 = u_t + 3u_x = \langle 1,3\rangle \cdot \langle u_t, u_x\rangle$$
$$= \langle 1,3\rangle \cdot \nabla u$$

that is $\nabla u = 0$ along any line.
parallel to $\langle 1,3\rangle$.

$\nabla u = 0$ means u is constant along
any line parallel to $\langle 1,3\rangle$,
which all have the form $\frac{dx}{dt} = 3$

$x = 3t + C$ or $x - 3t = C$

since u is constant along $x - 3t = C$, the value of u only depends on C.

so $u(t,x) = f(C) = f(x-3t)$.

for an arbitrary C^1 function f.

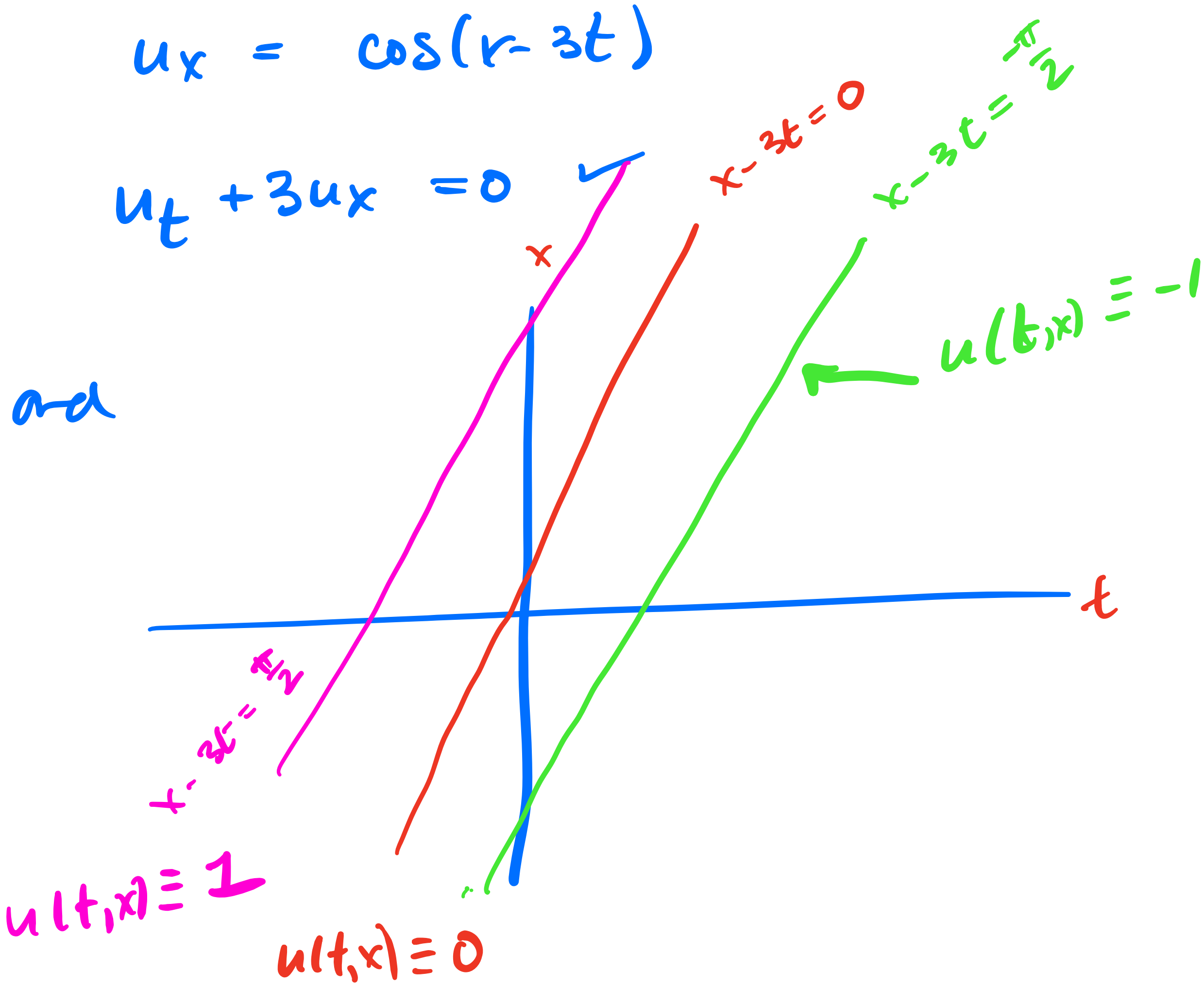
Example: $u(t,x) = \sin(x-3t)$

$u_t = \cos(x-3t)(-3)$

$u_x = \cos(x-3t)$

$u_t + 3u_x = 0$ ✓

and



these lines are called characteristic curves.

Another approach

Let $u_t + cu_x = 0$, c constant.

this is called the <u>transport equation</u>.

with wave speed c.

interpretation: $u(x,t)$ represents concentration (say of a pollutant) in a uniform fluid flow with velocity c.

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to solve we'll need an initial value,

say $u(0,x) = f(x)$ for $x \in \mathbb{R}$. f C^1.

The idea is to "make the equation stationary" by introducing a moving frame of reference moving at speed c along the flow.

If x is a position in the fixed frame of reference, let $\xi = x - ct$ represent the position of the object relative to an observer traveling the same velocity as the fluid.

Let us reformulate the problem in the eyes of the observer. by changing coordinates from (t,x) to (t,ξ).

Let $u(t,x) = v(t, x-ct) = v(t,\xi)$

where ξ is called the characteristic variable.

what PDE does v solve?

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} v(t,\xi) = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \xi}\cdot\frac{\partial \xi}{\partial t}$$

$$= \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} v(t,\xi) = \frac{\partial v}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} = \frac{\partial v}{\partial \xi}$$

$$u_t = v_t - c v_\xi$$

$$u_x = v_\xi.$$

if $u_t + c u_x = 0$

$$(v_t - c v_\xi) + c v_\xi = 0$$

$$\underline{v_t = 0}$$

we've reduced the problem to an ODE, one we can solve!

$$v(t, \xi) = f(\xi) = f(x - ct)$$
$$\parallel$$
$$u(t, x).$$

Example: $u_t + 3u_x = 0 \qquad u(x,0) = \dfrac{1}{1+x^2}$

has solution

$$u(t,x) = f(x - 3t)$$

$$u(0,x) = f(x) = \frac{1}{1+x^2}$$

identify arb funct.

so $u(t,x) = f(x-3t)$

$$= \frac{1}{1+(x-3t)^2}.$$

Prop 2.1 If $u(t,x)$ a solution to

$u_t + cu_x = 0$ defined on all of $\mathbb{R}^2$,

then

$u(t,x) = v(\xi)$ where

v is a C^1 function of $\xi = x - ct$.

Next bell and whistle:

Consider

$$u_t + cu_x + au = 0$$

most general first order linear constant coeff.

try the moving frame as before:

$$\xi = x - ct$$

$$u_t = v_t - cv_\xi$$

$$u_x = v_\xi$$

$$u_t + cu_x + au = 0$$

$v_t + av = 0$ First order linear "ODE"

let $\mu = e^{\int a\,dt} = e^{at}$

$e^{at} v_t + a e^{at} v = 0$

$\frac{\partial}{\partial t}\left(e^{at} v\right) = 0$

$e^{at} v = f(\xi)$

$v = e^{-at} f(\xi)$ f arb C^1.

in physical coords,

$u(t,x) = e^{-at} f(x - ct)$.

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if $a > 0$,

solutions move at speed c and decay at a rate determined by a.