A partial differential equation is an equation involving partial derivatives.

Ex: $\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x^2} = 0$

other standard notations:

- $u_{tt} + u_{xx} = 0$

- $\partial_t^2 u + \partial_x^2 u = 0$

A partial differential equation (or PDE) is solved by a function of more than one variable defined on a domain D which will typically be a connected open subset with a boundary ∂D.

The order of a PDE is the highest combined order of any derivative appearing.

Ex: $u_t = u_{xx}$ is second order.

A function $u(x_1, \dots x_n)$ is called smooth if it can be differentiated enough times.

More precisely, if a PDE has order n, we require a solution u to be class C^n, which means u and all derivatives up to order n are continuous on D.

These are called classical solutions

Ex: $u_t = u_{xx}$

one classical solution is

$$u(x,t) = t + \frac{1}{2}x^2$$

as $u_t = 1$

$u_{xx} = 1$

so $u_t = u_{xx}$.

other solutions:

$$u(t,x) = \frac{e^{-x^2/4t}}{2\sqrt{\pi t}}$$

$$u(t,x) = e^{-t}\cos x + i e^{-t}\sin x$$

normally, we'll require solutions to be physically meaningful.

First big difference from ODE:

an order n ODE has an n-dimensional solution space:

$$y_g = \underline{c_1} y_1 + \underline{c_2} y_2 + \dots + \underline{c_n} y_n.$$

n arbitrary constants

Solution to PDE

in general have solutions involving <u>arbitrary functions</u>.

you might have seen this before.

Suppose $u_{xy} = x + 2$

$$u_x = \int (x+2)\, \partial y$$

$$= xy + 2y + \underline{\underline{f(x)}}$$

$$u = \int xy + 2y + f(x)\, \partial x$$

$$= \frac{1}{2}x^2 y + 2xy + \underline{\underline{\int f(x)}} + \underline{\underline{g(y)}}$$

order 2, 2 arbitrary functions

upshot: PDEs have way more solutions than ODEs.

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vocabulary:

homogeneous linear:

the equation involves only sums of u or one of its derivaties to the first power.

Ex $u_t - u_{xx} = 0$ linear homogeneous.

nonhomogeneous linear:

the equation is linear but also involves a given function of the independent variables.

Ex: $u_t - u_{xx} = \cos(xt)$

A linear differential operator L satisfies

$$L(u+v) = L(u) + L(v)$$
$$L(cu) = cL(u)$$

A linear homogenous equation can be written

$L[u] = 0$ where L is a linear differential operator.

Ex: let $L = \frac{\partial^2}{\partial x^2}$

$$L[u+v] = \frac{\partial^2}{\partial x^2}(u+v) = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial x^2} v$$

$= L[u] + L[v].$

then $u_{xx} = 0$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$L[u] = 0$$

where $L = \frac{\partial^2}{\partial x^2} = \partial_x^2$

Ex:

$u_t - u_{xx} = 0$ (heat equation)

$$\frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = 0$$

$$(\partial_t - \partial_x^2) u = 0$$

$$L[u] = 0$$

where $L = \partial_t - \partial_x^2.$

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linear algebra idea:

nullspace A is a subspace:

if $A\vec{u}=0$ and $A\vec{v}=0$,

then $A(\vec{u}+\vec{v})=0$

and $A(c\vec{u})=0$

<u>principal of superposition</u>

if L is a liner operator

and $L[u]=0$ and $L[v]=0$

then $L[c_1 u + c_2 v]=0$.

More generally, if $u_1, \dots, u_n$ are solutions to $L[u]=0$, so is $\sum c_i u_i$

In liner algebra, every vector space is finite dimensional: we can find a basis for $\{\vec{v} : A\vec{v}=0\}$.

in PDE, the vector space

$\{u : L[u]=0\}$ will most often

be infinite dimensional, and we'll

need another approach to describe all solutions

An inhomogenous linear PDE has the form

$$L[u] = f$$ where f is known and L is a linear diff. operator.

Ex: $(\partial_t - \partial_{xx}) u = \underline{f(t,x)}$ forcing function.

if $u_1, \dots u_n$ are solutions to $Lu = 0$

and u^* is a particular solution to $Lu^* = f$

then $\sum_{i=1}^{n} c_i u_i + u^*$ is a solution to $Lu = f$ for any choice of c_i.

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why operators? sometimes we can use factoring ideas to reduce the complexity of a problem.

1D: $(D^2+4D+3)y=0$

$(D+3)(D+1)y=0$

either $(D+1)y=0$

$y'=-y \qquad y=e^{-t}$

or

$(D+1)(D+3)y=0$

so $(D+3)y=0$

$y'=-3y \qquad y=e^{-3t}$

2D:

$u_{xx}+4u_{xt}+3u_{tt}=0$

$(\partial_x^2+4\partial_x\partial_t+3\partial_t^2)u=0$

$(\partial_x+3\partial_t)(\partial_x+\partial_t)u=0$

$\vdots$