Two views of $\delta(x)$:

① $$\int_{\mathbb{R}} \delta(x-\xi) f(\xi)\, d\xi = f(x)$$

$$= \lim_{n\to\infty} \int_{\mathbb{R}} g_n(x-\xi) f(\xi)\, d\xi$$

② $$L_\xi[f] = f(\xi) = \langle \delta_\xi, f \rangle$$

$$= \lim_{n\to\infty} \langle g_n, f \rangle.$$

Let us integrate the delta function. How?

$$\int g_n(x)\,dx = \int \frac{n}{\pi(1+n^2x^2)}\,dx = \frac{1}{\pi}\tan^{-1}(nx) + C$$

Let us choose $C = \frac{1}{2}$ (this is a Fourier series choice)

$$\downarrow n\to\infty \qquad\qquad \downarrow n\to\infty$$

$$\int \delta(x)\,dx \qquad\qquad \sigma(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \\ 1/2 & x=0. \end{cases}$$

In some sense,

$$\int \delta(x)\,dx = \sigma(x)$$, and hence

$$\frac{d\sigma}{dx} = \delta.$$ (in line w/ FTC).

So we've built a notion of derivatives that extends to discontinuous functions.

The derivative $\frac{d\sigma}{dx} = \delta$ is called a <u>distributional derivative</u>.

Let us compute another example. If δ' exists, what should its formula be?

Let $0 \in (a,b)$.

$$\langle \delta', u \rangle = \int_a^b \delta'(x)\,u(x)\,dx$$

I'd love to get a δ in here. How? The single most important method of integration

$D = u(x) \qquad dI = \delta'(x)\,dx$

$dD = u'(x)\,dx \qquad I = \delta(x)$

$$= u(x)\,\delta(x)\Big|_a^b - \int_a^b \delta(x)\,u'(x)\,dx$$

$$= u(b)\,\delta(b) - u(a)\,\delta(a) - u'(0)$$

$$= 0 - 0 - u'(0)$$

$$= -u'(0).$$

hence: δ' is a distribution so that

$$\langle \delta', u \rangle = \langle \delta, -u' \rangle = -u'(0).$$

for all differentiable u.

<u>Note</u>: δ' means the derivative of δ as a distribution, not a function, so it only makes sense in the context of an integral. δ' is <u>not</u> a function derivative.

Def: Suppose $u \in L^1(a,b)$. $v \in L^1(a,b)$ is called a weak derivative of u if

$$\int_a^b u(x)\,\phi'(x)\,dx = -\int_a^b v(x)\,\phi(x)\,dx$$

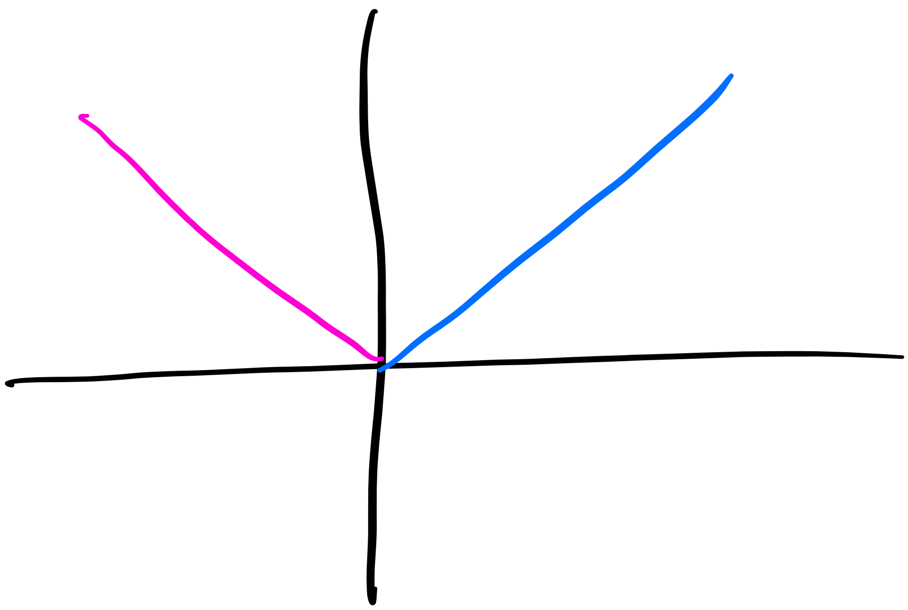
for all ϕ in $C_0^\infty(a,b)$

(smooth functions w/ $\phi(a) = \phi(b) = 0$.

Example:

Consider $u(x) = |x|$ on $(-1,1)$.

This is not a classically differentiable function.



Let ϕ be in $C_0^\infty(-1,1)$.

$$\int_{-1}^{1} |x|\,\phi'(x)\,dx = \int_{-1}^{0} (-x)\,\phi'(x)\,dx + \int_{0}^{1} x\,\phi'(x)\,dx$$

by parts

$u = -x \qquad dv = \phi' dx \qquad \Big| \qquad u = x \qquad dv = \phi' dx$

$du = -dx \qquad v = \phi \qquad \Big| \qquad du = dx \qquad v = \phi$

$$(-x)\phi(x)\Big|_{-1}^{0} + \int_{-1}^{0} \phi(x)\,dx + x\,\phi(x)\Big|_{0}^{1} - \int_{0}^{1} \phi(x)\,dx$$

0 as $\phi(-1)=0$ $\qquad$ 0 as $\phi(1)=0$

$$= (-1)\int_{-1}^{0} (-1)\,\phi(x)\,dx - \int_{0}^{1} (1)\,\phi(x)\,dx$$

$$= (-1)\int_{-1}^{1} \operatorname{sgn}(x)\,\phi(x)\,dx$$

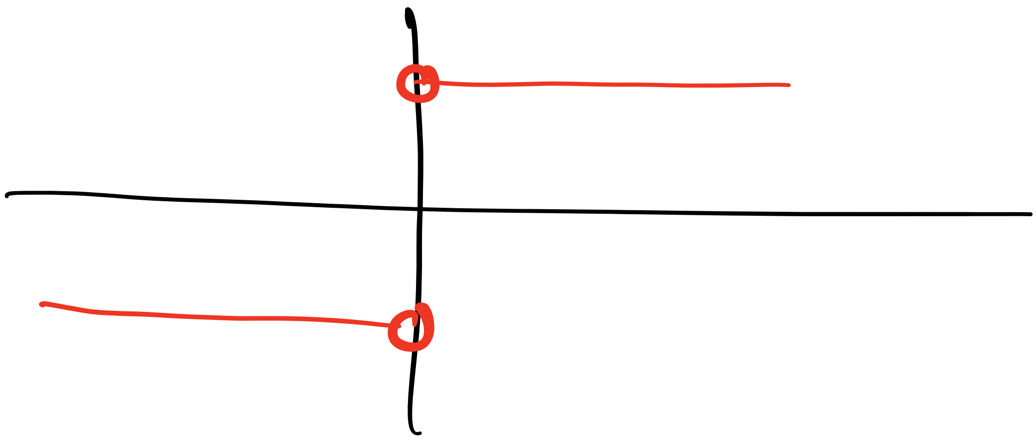
where $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$

$$\int_{-1}^{1} |x|\,\phi'(x)\,dx = -\int_{-1}^{1} \operatorname{sgn}(x)\,\phi(x)\,dx$$

so $\operatorname{sgn}(x)$ is the weak derivative of $|x|$.

$$\frac{d}{dx}|x| = \mathrm{sgn}(x).$$

$$\frac{d^2}{dx^2}|x| = \frac{d}{dx}\mathrm{sgn}(x) = 2\delta$$



So $$\frac{d^2}{dx^2}\left(\frac{1}{2}|x|\right) = \delta$$

ϕ is called a test function.

we found that

$\frac{d}{dx}|x| = \mathrm{sgn}(x)$ via testing by ϕ.

How does this lead to a method for evaluating differential equations?

consider

$$-u'' = \delta \qquad u(-1)=u(1)=0$$

test:

$$-u''\phi = \delta\phi \quad \text{for } \phi \in C_0^\infty(-1,1)$$

integrate:

$$\int_{-1}^{1} -u''\phi \, dx = \int_{-1}^{1} \delta(x)\phi(x) = \phi(0)$$

parts

$u = \phi \qquad dv = -u''$

$du = \phi' \qquad v = -u'$

$$\cancel{-u'\phi\Big|_{-1}^{1}} + \int_{-1}^{1} u'\phi' \, dx = \phi(0)$$

0

$$\int_{-1}^{1} u'\phi' \, dx = \phi(0).$$

we have the <u>weak</u> version of the equation.

find u with $u(1) = u(-1) = 0$ so

$$\int_{-1}^{1} u' \phi' \, dx = \phi(0) \text{ for all } \phi.$$

$$u(x) = \frac{1 - |x|}{2}$$

$$u'(x) = \frac{1}{2} - \frac{1}{2}\operatorname{sgn}(x) = \begin{cases} \frac{1}{2} & x < 0 \\ -\frac{1}{2} & x > 0 \end{cases}$$

$$\int_{-1}^{1} u' \phi' \, dx = \int_{-1}^{0} \tfrac{1}{2} \phi' + \int_{0}^{1} \left(-\tfrac{1}{2}\right) \phi'$$

$$= \tfrac{1}{2} \phi \Big|_{-1}^{0} - \tfrac{1}{2} \phi \Big|_{0}^{1}$$

$$= \tfrac{1}{2} \phi(0) + \tfrac{1}{2} \phi(0) = \underline{\phi(0)}$$

hence $u(x) = \dfrac{1 - |x|}{2}$ solves the weak problem

and hence is a weak solution to $-u'' = \delta$.

$$u' = -\frac{1}{2}\,\mathrm{sgn}(x)$$

$$u'' = -\frac{1}{2}(2\delta) = -\delta$$

wk derivatives