linear algebra parallel.

How do we understand a linear function $T(x) = Ax$? (A nonsingular).

Suppose we're trying to solve

$A\vec{x} = \vec{b}$.

One approach is to solve the smaller problems $A\vec{x} = \vec{e}_j$ where $\vec{e}_j$ is the j, The Dirac "function".

$$\delta(x) := \left.\begin{array}{ll} \delta(x) = 0 & x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)\,dx & = 1 \end{array}\right\} \text{incompatible}$$

$$\int_{-\infty}^{\infty} \delta(x) f(x) = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x-\xi) f(\xi)\,d\xi = f(x).$$

idea: introduce an impulse modeled by δ into a system and model its evolution by the PDE:

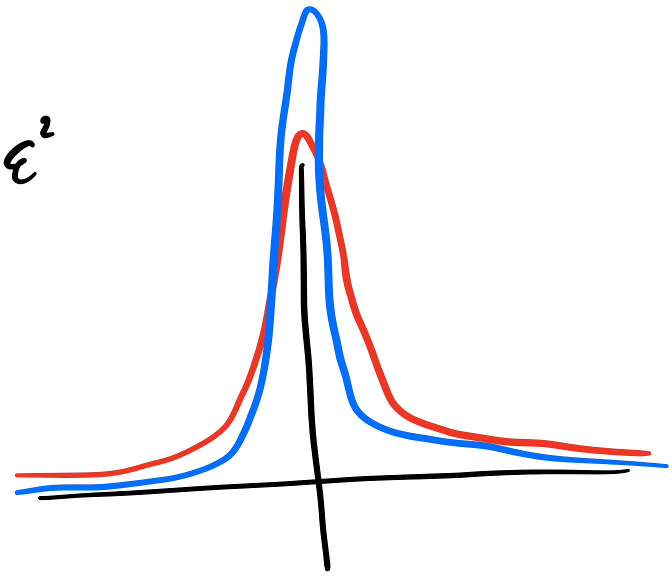
E.g.

$$u_t = u_{xx}$$

$$u(0,x) = \delta(x-\xi)$$

Lets look at some function families that "limit" to $\delta(x)$.

- $\delta_\varepsilon(x) = \dfrac{1}{\pi} \dfrac{\varepsilon}{x^2+\varepsilon^2}$



- $H_\varepsilon(x) = \dfrac{1}{\sqrt{\pi\varepsilon^2}} e^{-x^2/\varepsilon^2}$

In both cases, $\displaystyle\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \delta_\varepsilon(x) f(x)\, dx = f(0)$

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} H_\varepsilon(x) f(x)\, dx = f(0)$$

So in some sense, $\delta_\varepsilon \to \delta$, $H_\varepsilon \to \delta$ as $\varepsilon \to 0$.

If we cheat for a second, it almost looks like
$\langle \delta_\varepsilon, f \rangle \to f(0)$ $\quad \langle H_\varepsilon, f \rangle \to f(0)$

Our goal now is to incorporate forcing into our analysis.

we'll begin with

$$u_t = k u_{xx} + f(x,t), \text{ the forced heat equation.}$$

to start, lets take a new approach to

$$u_t = k u_{xx}$$
$$u(0,x) = f(x)$$
$$u(t,0) = u(t,l) = 0.$$

By superposition, if I knew the behavior of a concentrated unit bump at $\xi \in (0,l)$, I could build up behavior for a general f.

Basically, we want a function

$G(x,\xi,t)$ so that $G(x,\xi,0) = \delta(x-\xi)$

and $G_t = k G_{xx}$.

Then if I wanted the total response to $u(0,x) = f(x)$

$$u(t,x) = \int_0^\ell \boxed{G(x,\xi,t)}\, f(\xi)\, d\xi.$$

adding up a bunch of responses to an impulse of $f(\xi)$ at $x = \xi$.

In this case,

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{\ell}\int_0^\ell f(\xi)\sin\left(\frac{n\pi\xi}{\ell}\right) d\xi \; e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\frac{2}{\ell}\int_0^\ell \left(\sum_{n=1}^{\infty} f(\xi)\sin\left(\frac{n\pi\xi}{\ell}\right) e^{-k\left(\frac{n\pi}{\ell}\right)^2 t} \sin(n\pi x)\right) d\xi.$$

$$= \int_0^\ell \boxed{\left[\frac{2}{\ell}\sum_{n=1}^{\infty} e^{-k\left(\frac{\pi n}{\ell}\right)^2 t} \sin\left(\frac{n\pi\xi}{\ell}\right)\sin(n\pi x)\right]} f(\xi)\, d\xi.$$

$$G(x,\xi,t).$$

$$G(x,\xi,t) = \frac{2}{l}\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{l}\right)^2 t}\sin\left(\frac{n\pi\xi}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$u(t,x) = \int_0^l G(x,\xi,t)f(\xi)\,d\xi.$$

$$u(0,x) = \int_0^l \boxed{G(x,\xi,0)} f(\xi)\,d\xi = f(x)$$

$$\delta(x-\xi).$$

$$= \int_0^l \delta(x-\xi) f(\xi)\,d\xi = f(x).$$

so <u>formally</u>

$$G(x,\xi,0) \text{ “=” } \delta(x-\xi)$$

Dirac delta function

$G(x,\xi,t)$ is called the Green's function for the problem.