<u>Laplace's</u> <u>Equation</u> is a second order PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is perhaps the single most important PDE.

Solutions are called <u>harmonic</u> <u>functions</u>.

and are u in the kernel of the operator

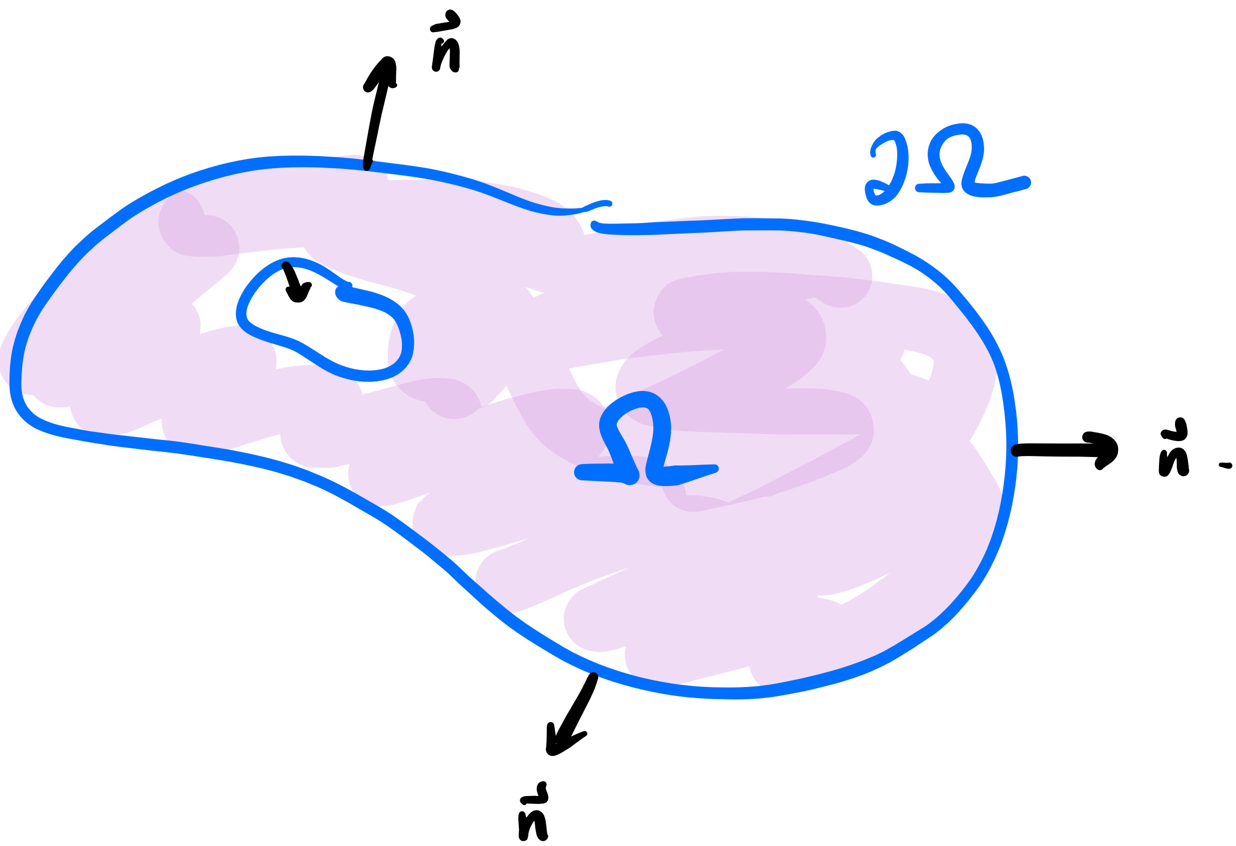
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$ called the

<u>Laplacian</u> (note: $\Delta = \nabla \cdot \nabla$)

The forced version

$$-\Delta u = f(x,y)$$ is called Poisson's equation.

Since there is no time dependence, these equations describe equilibrium and almost always appear in the context of boundary value problems.



what kind of bdy conditions?

$u(x,y) = h(x,y)$ for $(x,y) \in \partial\Omega$

<u>Dirichlet</u> (value on bdy)

$$\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n} = k(x,y) \qquad (x,y) \in \partial\Omega.$$

The <u>normal derivative</u> is specified.

typically, $\vec{n}$ is an outward facing unit normal

<u>Ex</u>: $\frac{\partial u}{\partial \vec{n}} = 0 \Rightarrow$ no flux through $\partial\Omega$.

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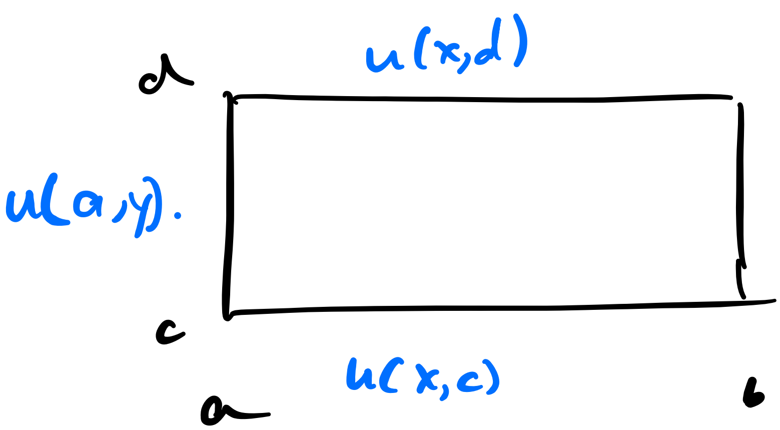
we'll consider two cases of Laplace's eqn where we can separate variables:

$u_{xx} + u_{yy} = 0$ on a rectangle



If $u(x,y) = X(x)Y(y)$,

then $X''Y + XY'' = 0$

$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

$$X'' \underbrace{- \lambda X = 0 \qquad Y'' + \lambda}_{\text{opposite signs!}} X = 0$$

If $\lambda = 0$

$X(x)$	$Y(y)$	u
$1, x$	$1, y$	$1, x, y, xy$

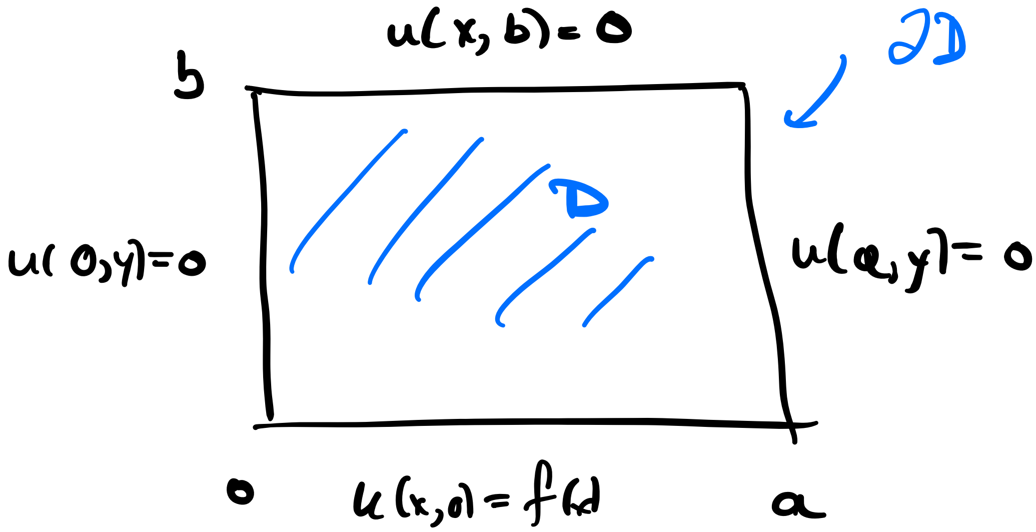
If $\lambda < 0$

$\cos(\mu x), \sin(\mu x)$ $\cosh(\mu y), \sinh(\mu y)$

If $\lambda > 0$

$\cosh(\mu x), \sinh(\mu x)$ $\cos(\mu y), \sin(\mu y)$.

consider



then $X''-\lambda X=0$ has

$$X(0)=X(a)=0$$

$$Y''+\lambda Y=0$$

$$Y(b)=0$$

separated boundary

<u>start w/ homogeneous solutions.</u>

$X''-\lambda X=0$

$X(0)=X(a)=0$ has

$\lambda=0$, $\lambda>0$ impossible,

so $\lambda<0$, $\lambda=-\mu^2$.

$$X(x)=A\cos(\mu x)+B\sin(\mu x)$$

$X(0) = A = 0$

$X(a) = B\sin(\mu a) = 0$ so $\mu = \frac{n\pi}{a}$ $n = 1, \ldots$

$X_n(x) = \sin\left(\frac{n\pi}{a}x\right)$

$\lambda = -\mu^2 = \frac{-n^2\pi^2}{a^2}$

so $Y'' + \lambda Y = 0$ becomes

$Y'' - \frac{n^2\pi^2}{a^2} Y = 0$ $n = 1, 2, \ldots$

$Y = c_1 e^{\frac{n\pi}{a}y} + c_2 e^{-\frac{n\pi}{a}y}$

$Y(b) = 0$

$c_1 e^{\frac{n\pi b}{a}} + c_2 e^{-\frac{n\pi b}{a}} = 0$

$\Rightarrow c_2 = -c_1 e^{\frac{2\pi n b}{a}}$

$Y = c_1 e^{\frac{n\pi}{a}y} - c_1 e^{\frac{n\pi b 2}{a}} e^{-\frac{n\pi}{a}y}$

$= c_1 e^{\frac{n\pi b}{a}} \left(e^{-\frac{n\pi b}{a}} e^{\frac{n\pi}{a}y} - e^{\frac{n\pi b}{a}} e^{-\frac{n\pi y}{a}} \right)$

$$= 2c_1 e^{\frac{n\pi b}{a}} \left(\frac{-e^{-\frac{n\pi}{a}(b-y)} + e^{\frac{n\pi(b-y)}{a}}}{2} \right)$$

$$= 2c_1 e^{\frac{n\pi b}{a}} \underline{\sinh\left(\frac{n\pi}{a}(b-y)\right)}$$

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so $u_n = \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}(b-y)\right)$ $n=1,2\ldots$

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}(b-y)\right)$$

can we find c_n?

$$u(x,0) = f(x)$$

$$= \sum_{n=1}^{\infty} \underbrace{\boxed{c_n \sinh\left(\frac{n\pi b}{a}\right)}}_{b_n.} \sin\left(\frac{n\pi}{a}x\right)$$

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

so

$$c_n = \frac{b_n}{\sinh\left(\frac{n\pi b}{a}\right)}$$

so

$$u(x,y) = \sum_{n=1}^{\infty} \frac{b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

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polar version:

suppose we're interested in a circular domain.

lets use polar coordinates

$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2+y^2}$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial}{\partial x} u(r,\theta) = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r} \cdot \frac{2x}{2\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta} \cdot \frac{1}{1+(y/x)^2} \cdot \frac{-y}{x^2}$$

$$= \frac{\partial u}{\partial r} \cdot \frac{2r\cos\theta}{2r} + \frac{\partial u}{\partial \theta} \cdot \frac{-r\sin\theta}{r^2}$$

$$= \left(\frac{\partial}{\partial r} \cdot \cos\theta + \frac{\partial}{\partial \theta} \frac{\sin\theta}{r}\right) u.$$

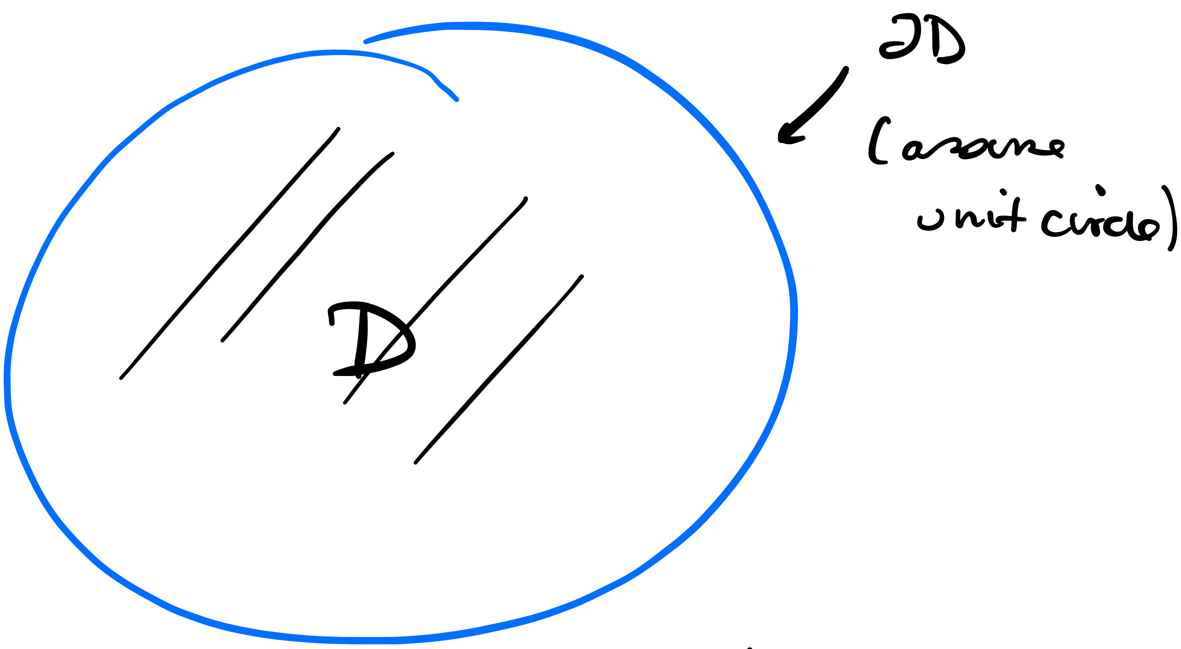
$$\frac{\partial}{\partial x} = \cos\theta\, \partial_r - \frac{\sin\theta}{r}\partial_\theta .$$

easy to compute the rest.

plugging into $u_{xx}+u_{yy}=0$

$\Rightarrow$ Polar form of Laplace's eq^n.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$



$u(1,\theta) = h(\theta)$

Dirichlet boundary conditi

$$u = R(r)\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

since $r \neq 0$ in polar,

$$\frac{R''}{R} + \frac{1}{r}\cdot\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

so $r^2R'' + rR' - \lambda R = 0$

and

$$\Theta'' + \lambda\Theta = 0$$

we should impose periodic boundary conditions

$\Theta(0) = \Theta(2\pi)$

$\Theta'(0) = \Theta'(2\pi)$

we've already solved this:

the eigenfunctions are $1, \sin n\theta, \cos n\theta$

for $n = 1, 2, \ldots$

for $\lambda = n^2$.

Applying this to the R equation gives

$$r^2 R'' + r R' - n^2 R = 0$$

this is called a Cauchy-Euler equation,

which has ansatz $R = r^k$

$R' = k r^{k-1}$

$R'' = k(k-1) r^{k-2}$

$r^2 R'' + r R' - n^2 R = 0$ becomes

$$(k(k-1) + k - n^2) r^k = 0$$

iff

$$(k^2 - n^2) r^k = 0$$

iff

$$k^2 - n^2 = 0 \qquad k = \pm n.$$

so for $n \neq 0$, $R = r^n, r^{-n}$ are eigensolutions

for $n=0$,

$$r^2 R'' + rR' = 0$$

is solved by $R=1$ and $R = \ln r$

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so the full eigensolutions are

1, $\ln r$, $r^n \cos n\theta$, $r^n \sin n\theta$, $r^{-n} \cos n\theta$, $r^{-n} \sin n\theta$.

unbounded as $r \to 0$.

hidden boundary condition: $u(r,\theta)$ is bounded on D.

$$\Rightarrow \quad u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta \right)$$

$$\underset{\substack{\parallel \\ h(\theta)}}{u(1,\theta)} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

so $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta \, d\theta$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin n\theta \, d\theta$$

Note: $r^n\cos n\theta$ and $r^n\sin n\theta$ are called the <u>harmonic</u> polynomials