1D wave equation

Before, we solved the 1D wave equation w/ no boundary.

Let us apply separation of variables to the wave equation.

Let $u(t,x) = w(t)\,v(x)$

$$u_{tt} = c^2 u_{xx}$$

$$w'' v = c^2 w v''$$

$$\frac{w''}{w} = c^2 \frac{v''}{v} = \lambda$$

$$\frac{w''}{w} = \lambda \qquad \frac{c^2 v''}{v} = \lambda$$

$$w'' - \lambda w = 0 \qquad v'' - \frac{\lambda}{c^2} v = 0$$

Separable solutions

λ	$w(t)$	$v(x)$	$u(t,x)$
0	$1, t$	$1, x$	$1, x, t, xt$
$\lambda = -\omega^2 < 0$	$\cos\omega t, \sin\omega t$	$\cos\frac{\omega x}{c}, \sin\frac{\omega x}{c}$	$\cos\omega t\cos\frac{\omega x}{c}$, $\cos\omega t\sin\frac{\omega x}{c}$, $\sin\omega t\cos\frac{\omega x}{c}$, $\sin\omega t\sin\frac{\omega x}{c}$
$\lambda = \omega^2 > 0$	$e^{-\omega t}, e^{\omega t}$ or cosh, sinh.	$e^{-\omega x/c}, e^{+\omega x/c}$	$e^{-\omega(t+x/c)}, e^{\omega(t+x/c)}$, $e^{-\omega(t-x/c)}, e^{\omega(t-x/c)}$

$u(x,t)$ = displacement from equilibrium at (t,x).

Let us fix a string at ends $x=0$, $x=\ell$.

$u(t,0) = u(t,\ell) = 0$

we also give initial conditions

$u(0,x) = f(x)$ displacement

$u_t(0,x) = g(x)$ velocity

just like the d'Alembert case.

Notice that

$u(t,0) = u(t,\ell)$ w/ $u(t,x) = w(t)\,v(x)$

so $\underline{v(0) = v(\ell)}$

and we're solving

$$v'' - \frac{\lambda}{c^2} v = 0 \qquad v(0) = v(\ell)$$

$$\Rightarrow \quad v_n(x) = \sin\left(\frac{n\pi x}{\ell}\right) \qquad \lambda = -\left(\frac{n\pi c}{\ell}\right)^2 \quad n=1,2\ldots$$

so

$$u_n(t,x) = \cos\left(\frac{n\pi ct}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\tilde{u}_n(t,x) = \sin\left(\frac{n\pi ct}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$u(t,x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi ct}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right) + d_n \sin\left(\frac{n\pi ct}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$u(0,x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) = f(x)$$

so $b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$

$$u_t(t,x) = \sum_1^\infty b_n\left(-\frac{n\pi c}{\ell}\right)\sin\left(\frac{n\pi ct}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right) + d_n\left(\frac{n\pi c}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$u_t(0,x) = \sum_1^\infty \underbrace{d_n\left(\frac{n\pi c}{\ell}\right)}\sin\left(\frac{n\pi x}{\ell}\right) = g(x)$$

$$d_n\left(\frac{n\pi c}{\ell}\right) = \frac{2}{\ell}\int_0^\ell g(x)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$d_n = \frac{2}{n\pi c}\int_0^\ell g(x)\sin\left(\frac{n\pi x}{\ell}\right)$$

$u_{tt} = u_{xx}$

Example: string from 0 to 1.

$u_t(0,x) = 0$ $\quad$ $u(0,x) = 2x$ on $(0,\frac{1}{2})$

$2-2x$ on $(\frac{1}{2},1)$.

<u>$u(t,0) = u(t,1) = 0.$</u>

$$d_n = 0$$

$$b_n = \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

$$u(t,x) = \sum_{n=1}^{\infty}\left(\frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\right)\cos(n\pi t)\sin(n\pi x)$$