Let's revisit the heated ring.

$u_t = \gamma u_{xx}$

$u(t,-\pi) = u(t,\pi)$
$u_x(t,-\pi) = u_x(t,\pi)$ } periodic boundary conditions

$u(0,x) = f(x) \in L^2$

$\lambda = 0$ $\quad u = ax+b$

$\lambda < 0$ $\quad u = e^{\lambda\gamma t}(A\cos(\mu x) + B\sin(\mu x))$

$-\mu^2 = \lambda$

$\lambda > 0$ $\quad u = e^{\lambda\gamma t}(A\cosh(\mu x) + B\sinh(\mu x))$

$\mu^2 = \lambda$

$\lambda = 0$

$u(t,-\pi) = u(t,\pi) \Rightarrow \quad a\pi + b = -a\pi + b \Rightarrow a = 0$

$u_x(t,-\pi) = u_x(t,\pi) \Rightarrow \quad a = a$ no new info.

so $\lambda = 0$ is an eigenvalue with

$u(t,x) = b$ the eigensolution

(in fact the equilibrium solution)

$\lambda > 0$

$$u = e^{\lambda \gamma t}\left(A\cosh(\mu x) + B\sinh(\mu x)\right)$$

$$u_x = e^{\lambda \gamma t}\left(A\mu\sinh(\mu x) + B\mu\cosh(\mu x)\right)$$

<u>IC:</u>

$u(t,-\pi) = u(t,\pi) \Rightarrow$

$$\overset{\text{even}}{A\cosh(\mu\pi)} + \overset{\text{odd}}{B\sinh(\mu\pi)}$$

$$= A\cos(-\mu\pi) + B\sinh(-\mu\pi)$$

so $\cancel{A\cos(\mu\pi)} + B\sinh(\mu\pi) = \cancel{A\cosh(\mu\pi)} - B\sinh(\mu\pi)$

$$2B\sinh(\mu\pi) = 0$$

$$\mu \neq 0,$$

so $B = 0$.

then $u_x = e^{\lambda \gamma t}\left(A\mu\sinh(\mu x)\right)$

$u_x(t,-\pi) = u_x(t,\pi) \Rightarrow A\mu\sinh(-\mu\pi) = A\mu\sinh(\mu\pi)$

$$0 = 2A\mu\sinh(\mu\pi)$$

$\mu \neq 0$, so $A = 0$.

so $\lambda > 0$ is <u>not</u>

an eigenvalue.

$\lambda < 0$ $\quad u = e^{\lambda \gamma t}\left(A\cos(\mu x) + B\sin(\mu x)\right)$

$$u_x = e^{\lambda \gamma t}\left(-\mu A\sin(\mu x) + \mu B\cos(\mu x)\right)$$

$u(t,\pi) = u(t,-\pi) \Rightarrow$

$$A\cos(\mu\pi) + B\sin(\mu\pi) = A\cos(-\mu\pi) + B\sin(-\mu\pi)$$

$$\Rightarrow \quad 2B\sin(\mu\pi) = 0$$

$\sin(\mu\pi) = 0$ if $\mu = 1, 2, 3, \ldots$

$u_x(t,-\pi) = u_x(t,\pi) \Rightarrow$

$$-\mu A\sin(\mu\pi) + \mu B\cancel{\cos(\mu\pi)} = -\mu A\sin(-\mu\pi) + \mu B\cancel{\cos(\mu\pi)}$$

$$2\mu A\sin(\mu\pi) = 0$$

if $\mu = 1, 2, 3, \ldots$

so $\lambda = -\mu^2 = -1, -4, -9, -16, -25, \ldots$ are eigenvalues

$$u(t,x) = e^{-n^2 t}\left(A_n\cos(nx) + B_n\sin(nx)\right)$$

are eigensolutions.

$$u(t,x) = b + \sum_{n=1}^{\infty} e^{-n^2 t}\left(A_n \cos nx + B_n \sin(nx)\right)$$

$$u(0,x) = b + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin(nx) = \underline{f(x)}$$

so $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx$

$b = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx.$

Neumann Conditions

Now let us consider the case where the ends are insulated: no diffusion across the boundary.

We require that the normal derivative be 0 in D, this is

$u_x(t,0) = 0 \qquad u_x(t,l)=0$

for a rod of length l,

$$u_t = \gamma u_{xx} \qquad u(0,x) = f(x).$$
$$u_x(t,0)=0$$
$$u_x(t,l)=0.$$

Aagain applying $u = w(t)v(x)$,

we find $\lambda=0$ is a eigenvalue w

$u(t,x) = b$

$\lambda > 0$ is not an eigenvalue,

and $\lambda < 0$ <u>is</u> an eigenvalue with $\lambda = -n^2$, $\mu = n$

$$\underline{u(t,x) = e^{-\frac{n^2\pi^2}{\ell^2}t} \cos\left(\frac{n\pi x}{\ell}\right)}$$

so our general seperable solution is

$$u(t,x) = b + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{\ell^2}t} \cos\left(\frac{n\pi x}{\ell}\right)$$

Heat equation w/ dissipation:

$$u_t = u_{xx} - \underline{\underline{2u}}$$

$u_t = -\alpha(u - u_0)$ is Newton's law of cooling.

Heat dissipates at a rate proportional to between temp and ambient temp.

$$w'v = wv'' - 2vw$$

$$\frac{w'}{w} + 2 = \frac{v''}{v} = \lambda$$

$$\frac{w'}{w} + 2 = \lambda \qquad \frac{v''}{v} = \lambda$$

$$w' + (2-\lambda)w = 0 \qquad v'' - \lambda v = 0$$

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impose $u_x(t,0) = u_x(t,\ell) = 0$ for all t.

$\Rightarrow \lambda = 0$ has $v = b$, $w = e^{-(2-\lambda)t} = e^{-2t}$

$$u = e^{-2t}(b)$$

$\lambda = 0$

$\lambda > 0$ physically impossible.

$\lambda < 0$

$$\lambda = -\frac{n^2\pi^2}{\ell^2}$$

$$v = \cos\left(\frac{n\pi x}{\ell}\right)$$

$$w = e^{\left(-2-\frac{n^2\pi^2}{\ell^2}\right)t}$$

$$u = e^{-2t}\cdot b + \sum_{n=1}^{\infty} a_k\, e^{\left(-2-\frac{n^2\pi^2}{\ell^2}\right)t} \cos\left(\frac{n\pi x}{\ell}\right)$$

$u \to 0$ as $t \to \infty$.

$$a_k = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right) dx$$

$$b = \frac{1}{\ell}\int_0^{\ell} f(x)\, dx.$$