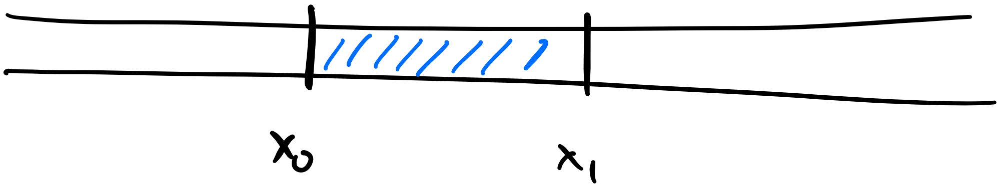

$u(x,t)$ = mass concentration of dye at time t, position x.

$M(t) = \int_{x_0}^{x_1} u(t,x)\,dx$ = mass of dye between x_0 and x_1 at time t.

$$\frac{d}{dt} M(t) = \frac{d}{dt} \int_{x_0}^{x_1} u(t,x)\,dx$$

$$= \int_{x_0}^{x_1} \frac{\partial}{\partial t} u(t,x)\,dx$$

$$M'(t) = \int_{x_0}^{x_1} u_t(t,x)\,dx$$

observation: dye moves at a rate proportional to concentration gradient.

in the tube

$$\frac{dM}{dt} =$$

$$= k\big(\underbrace{u_x(x_1,t) - u_x(x_0,t)}_{\text{concentration gradient.}}\big)$$

so $\int_{x_0}^{x_1} u_t(t,x)\,dx = \frac{dM}{dt} = k\big(u_x(x_1,t) - u_x(x_0,t)\big)$

$$\frac{\partial}{\partial x_1} \int_{x_0}^{x_1} u_t(t,x)\,dx = \frac{\partial}{\partial x_1} k\left(u_x(x_1,t) - u_x(x_0,t)\right)$$

$$u_t(t,x_1) = k\, u_{xx_1}(x_1,t)$$

so $\boxed{u_t(t,x) = k u_{xx}(t,x)}$

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the <u>heat equation</u> in 1D is

$$u_t = \gamma u_{xx}. \qquad \underline{\gamma > 0}$$

γ is called <u>thermal diffusivity</u>.

we've looked at this equation w/ the ansatz $u = e^{-\lambda t} v(x)$. where does this come from?

A solution to a PDE is called <u>seperable</u>

if $u = w(t) v(x)$.

If $u_t = \gamma u_{xx}$, then

$$w'(t) v(x) = \gamma w(t) v''(x)$$

$$\text{so } \frac{w'(t)}{\gamma w(t)} = \frac{v''(x)}{v(x)}$$

If a function of t is going to everywhere be equal to a function of x, both functions must be constant

$$F(t,x) = \frac{w'(t)}{w(t)} = \gamma \frac{v''(x)}{v(x)}$$

$$\frac{\partial}{\partial t} F = \frac{\partial}{\partial t} \gamma \frac{v''(x)}{v(x)} = 0$$

$$\frac{\partial}{\partial x} F = \frac{\partial}{\partial x} \frac{w'(t)}{w(t)} = 0$$

so F must be constant

so $$\frac{w'(t)}{\gamma w(t)} = \frac{v''(x)}{v(x)} = \lambda$$

So $\dfrac{w'(t)}{\gamma w(t)} = \lambda$ and $\dfrac{v''(x)}{v(x)} = \lambda$

$w'(t) = \lambda \gamma w(t)$ and $v''(x) = \lambda v(x)$.

suppose we have a beam of length ℓ,

$u(t,\ell) = 0 \quad u(t,0) = 0 \quad t \geq 0$

$u(0,x) = f(x) \quad 0 \leq x \leq \ell$.

Dirichlet boundary conditions: specify temp at ends.

Sln $v''(x) - \lambda v(x) = 0 \quad v(0) = 0 \quad v(l) = 0$

if $\lambda = 0$,

$v'' = 0$

$v = ax + b \qquad v(0) = 0 \Rightarrow b = 0$

$v(l) = 0 \rightarrow a = 0$

no solution.

if $\lambda = \mu^2 > 0$,

$v'' - \mu^2 v = 0$

$\Rightarrow \quad v = c_1 \cos \mu x + c_2 \sinh \mu x$

$v(0) = c_1 \cosh(0) + c_2 \sinh(0)$

$= \underline{c_1 = 0}$

$v(l) = c_2 \sinh(\mu l) = 0$ only if $\underline{c_2 = 0}$

so $\underline{c_1, c_2 = 0}$

if $\lambda = -\mu^2 < 0$,

$v'' + \mu^2 v = 0$

$\rightarrow$

$v = c_1 \cos \mu x + c_2 \sin \mu x$

$v(0)=0 \Rightarrow c_1=0$

$v(\ell)=0 \Rightarrow c_2 \sin \mu\ell = 0$

$\Rightarrow \mu\ell = n\pi \quad n=1,2,3,\dots$

$\Rightarrow \mu = \frac{n\pi}{\ell}$

$\Rightarrow \lambda = \frac{n^2\pi^2}{\ell^2}.$

so $v(x) = \sin\left(\frac{n\pi}{\ell}x\right) \quad n=1,2,\dots$

$$u(t,x) = e^{\gamma\lambda t} \sin\left(\frac{n\pi}{\ell}x\right)$$

$$= e^{-\frac{\gamma n^2\pi^2}{\ell^2}t} \sin\left(\frac{n\pi}{\ell}x\right).$$

then the general separable solution is

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{\frac{-\gamma n^2\pi^2 t}{\ell^2}} \sin\left(\frac{n\pi}{\ell}x\right).$$

what about the b_n?

$$u(0,x) = f(x) = \sum_{n=1}^{\infty} \underbrace{b_n \sin\left(\frac{n\pi}{\ell}x\right)}_{\text{this is a Fourier series!}}$$

$$b_n = \langle f, \sin\left(\tfrac{n\pi}{\ell}x\right)\rangle$$

$$= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx.$$

$$= \frac{2}{\ell}\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx \qquad \text{since } f \text{ must be odd.}$$

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what if the Dirichlet boundary conditions aren't homogeneous?

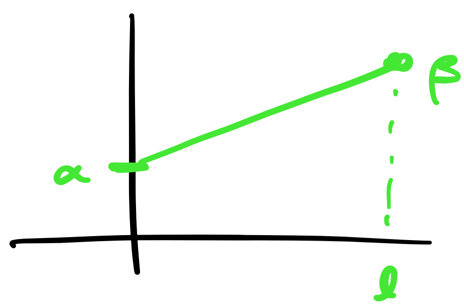
Say $\quad u(t,0)=\alpha \qquad u(t,\ell)=\beta.$

$$u_t = D u_{xx} \qquad t \geq 0.$$

what should happen in the long run?

the equilibrium solution will be

$$u^*(x) = \alpha + \frac{\beta-\alpha}{\ell}x$$



if we remove the equilibrium solution,

we can solve the equation describing deviation from equilibrium.

$$\tilde{u}(t,x) = u(t,x) - u^*(x)$$

$$\tilde{u}(t,0) = 0 = \tilde{u}(t,\ell)$$

$$\tilde{u}(0,x) = u(0,x) - u^*(x) = f(x) - \alpha + \frac{\beta-\alpha}{\ell}x \equiv \tilde{f}(x).$$

then

$$\tilde{u}(t,x) = \sum_{k=1}^{\ell} \tilde{b}_k e^{-\gamma \frac{n^2\pi^2}{\ell^2} t} \sin\left(\frac{k\pi x}{\ell}\right)$$

where $\tilde{b}_k = \frac{2}{\ell}\int_0^{\ell} \tilde{f}(x) \sin\left(\frac{k\pi x}{\ell}\right)$

so $$u(t,x) = \alpha + \frac{\beta-\alpha}{\ell}x + \underbrace{\sum_{k=1}^{\ell} \tilde{b}_k e^{-\gamma \frac{n^2\pi}{\ell^2} t} \sin\left(\frac{k\pi x}{\ell}\right)}_{\text{decays as exponential w/ } \frac{n^2}{\ell^2}}$$

A bit about convergence to equilibrium.