This is a course in partial differential equations. The primary tools will be techniques from ordinary differential equations.

So we should review:

First order linear equations:

standard form

$$y' + p(x)y = q(x).$$

Method of solution:

$\boxed{\text{integrating factor}}$

$$\mu = e^{\int p(x)}$$

$$\underbrace{e^{\int p} y' + e^{\int p} p(x) y}_{\text{product rule}} = e^{\int p} q(x)$$

$$\frac{d}{dx}\left(e^{\int p} y\right) = e^{\int p} q(x)$$

$$e^{\int p} y = \int e^{\int p} q(x) + C$$

$$y = \frac{1}{e^{\int p}} \left(\int e^{\int p} q(x) + C \right).$$

<u>Separable</u> <u>equations</u>

standard form:

$$y' = F(x)\, G(y)$$

method of solution: separation

$$\frac{y'}{G(y)} = F(x)$$

$$\frac{dy}{G(y)} = F(x)\, dx$$

$$\int \frac{dy}{G(y)} = \int F(x)\, dx$$ implicitly solves equation.

Second (or higher) order linear homogeneous constant coefficient equations

Standard form:

$$Ay'' + By' + Cy = 0$$

Method of solution:

test solution $y = e^{\lambda t}$

$$y' = \lambda e^{\lambda t}$$

$$y'' = \lambda^2 e^{\lambda t}$$

$$A\lambda^2 e^{\lambda t} + B\lambda e^{\lambda t} + Ce^{\lambda t} = 0$$

$$(A\lambda^2 + B\lambda + C)\, e^{\lambda t} = 0$$

$$\underbrace{A\lambda^2 + B\lambda + C}_{} = 0$$

characteristic polynomial. $P(\lambda)$

roots give solutions.

If $\lambda_1 \neq \lambda_2$ and $p(\lambda_1) = 0$
$p(\lambda_2) = 0$,

$$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

If $\lambda_1 = \lambda_2$

$$y = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$$

Operator perspective:

let $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$.

$$y'' - 3y' + 2y = 0$$

$$(D^2 - 3D + 2)y = 0$$

$$L[y] = 0$$

where $L = D^2 - 3D + 2$.

we're looking for vectors y so

y is in the kernel or nullspace of L.

how could this happen?

$$(D-2)\underbrace{(D-1)y}=0$$

$$(D-1)\underbrace{(D-2)y}=0$$

$y' - y = 0$ $\quad y = e^{t}$

$y' - 2y = 0$ $\quad y = e^{2t}$

so the kernel of L is the span of
e^t and e^{2t},
$$y = C_1 e^t + C_2 e^{2t}$$

nonhomogeneous second (or higher) order constant coefficient linear equations.

standard form:
$$Ay'' + By' + Cy = f(t)$$

method of solution:

first, find everything $L = AD^2 + BD + C$ sends to 0 by solving
$$Ay'' + By' + Cy = 0$$
$$(A\lambda^2 + B\lambda + C) = 0$$
$$\Rightarrow y_h = C_1 e^{\lambda_1 t} + C_2^{\lambda_2 t}$$

then solve
$$Ay'' + By' + Cy = f(x)$$ for a particular solution

by

- method of undetermined coefficients
  (guess and check)

- annihilators

Ex: $y'' - 3y' + 2y = t^2 + 1$

1. $y'' - 3y' + 2y = 0$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$y_p = c_1 e^{2t} + c_2 e^{t}$$

2. particular.

guess $y_p = at^2 + bt + c$

$$y_p' = at + b$$

$$y_p'' = a$$

$$(a) - 3(at + b) + 2(at^2 + bt + c) = t^2 + 1$$

$$a - 3ax - 3b + 2ax^2 + 2bx + 2c = t^2 + 1$$

$$(2a)t^2 + (-3a + 2b)t + (a - 3b + 2c) = t^2 + 1$$

$2a = 1 \quad \Rightarrow \quad a = \frac{1}{2}$

$-3a + 2b = 0 \qquad\qquad -\frac{3}{2} + 2b = 0$

$$b = \frac{3}{4}$$

$$\frac{1}{2} - 3\left(\frac{3}{4}\right) + 2c = 1$$

$$2c = 1 - \frac{1}{2} + \frac{9}{4}$$

$$c = \frac{11}{8}.$$

$$y_p = \frac{1}{2}t^2 + \frac{3}{4}t + \frac{11}{8}.$$

So $$y = \underbrace{c_1 e^{2t} + c_2 e^{t}}_{\text{in ker } L} + \underbrace{\frac{1}{2}t^2 + \frac{3}{4}t + \frac{11}{8}}_{\text{in ran } L.}$$

Second order linear DE:

$$y'' + p(x)y' + q(x)y = f(x)$$

if one solution is known, we can often find another via <u>reduction of order</u>.

Given a solution $y_1(t)$,

Guess $y_2(t) = v(t)\, y_1(t)$ and find $v(t)$.

$$y'' + 4y' + 4y = 0$$

$$(\lambda^2 + 4\lambda + 4) = 0$$

$$(\lambda + 2)^2 = 0 \qquad \text{so} \quad y_1(t) = e^{-2t}$$

Guess
$$4y_2 = \underline{\cancel{4v(t)e^{-2t}}}$$
$$4y_2' = \cancel{4v'(t)}e^{-2t} \; \underline{\cancel{-4\cdot 2v(t)e^{-2t}}}$$
$$y_2'' = v''(t)e^{-2t} \underline{\cancel{-2v'(t)e^{-2t}}} \underline{\cancel{-2v'(t)e^{-2t}}}$$
$$\underline{\cancel{+4v(t)e^{-2t}}}$$

$$y_2'' + 4y' + 4y$$
$$= v''(t)e^{-2t} = 0$$

$$v'' e^{-2t} = 0$$

so
$$v'' = 0$$
$$v' = C$$
$$v = Ct \qquad \text{choose } C = 1$$

$$v = t$$
$$y_2(t) = te^{-2t}$$

so $y = C_1 e^{-2t} + C_2 t e^{-2t}$.

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from second order to system

$$y'' + p(t)y' + q(t)y = 0$$

let $u_1 = y$

$u_2 = y'$

then. $u_1' = u_2$

$$u_2' + p u_2 + q u_1 = 0$$

$$u_1' = u_2$$

$$u_2' = -p u_2 + q u_1$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

system form