**Name:** _____

**Math 418 (Tully-Doyle)** **Final Exam** **April 21, 2026**

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*Note:* **No notes. No calculators.** *Show all your work and clearly indicate your answer for full credit. Partial solutions will receive partial credit.*

| Question | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 10 | |
| 5 | 15 | |
| 6 | 10 | |
| 7 | 5 | |
| 8 | 5 | |
| Total: | 60 | |

Initials:

1. (5 points) Find the solution $u = u(x, y)$ to the PDE

$$u_x - 4xy^2\, u_y = 0$$
$$u(1, y) = y.$$

Initials:

2. (5 points) (a) Find the characteristic change of variables for the equation

$$2u_x + 5u_y - u = 1.$$

(b) Use the change of variables from part (a) to solve the PDE.

Initials:

3. (5 points) Consider the initial value problem for $x \in (-\infty, \infty)$ given by

$$\begin{aligned} DE: &\quad u_{tt} = 9u_{xx}, \\ IC: &\quad u(x,0) = e^{-x}; \\ &\quad u_t(x,0) = \cos x. \end{aligned}$$

Find $u(x,t)$. Verify that it is a solution to the equation.

Initials:

4. (10 points) Consider the initial value problem for $x \in (-\infty, \infty), t > 0$ given by

$$\begin{aligned} DE: &\quad u_t = u_{xx} - u, \\ IC: &\quad u(x,0) = f(x). \end{aligned}$$

Solve for $u(x,t)$. (Hint: use the substitution $u = e^{-t}v$)

Initials:

5. Consider the boundary value problem

$$\begin{aligned} DE: &\quad u_t = u_{xx} \\ BC: &\quad u(0,t) = u(\pi,t) = 0 \\ IC: &\quad u(x,0) = f(x). \end{aligned}$$

(a) (2 points) Use separation of variables to write a system of separated equations in $X, T$.

(b) (8 points) Find the eigenvalues and corresponding eigenfunctions.

(part (b) continued)

Initials:

(c) (5 points) Solve the related system with constant dissipation

$$\begin{aligned} DE: &\quad u_t = u_{xx} + u \\ BC: &\quad u(0,t) = u(\pi,t) = 0 \\ IC: &\quad u(x,0) = f(x). \end{aligned}$$

How does the steady state behavior compare to that of your solution in part (b)?

Initials:

6. (10 points) Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in [\pi/2, \pi) \\ 0 & \text{if } x \in (-\pi/2, \pi/2) \\ -1 & \text{if } x \in (-\pi, \pi/2] \end{cases}.$$

(a) Find a Fourier series expansion for $f$.

(b) What does the Fourier series converge to at $x = \pi/2$? How do you know?

Initials:

7. (5 points) Suppose that

$$u(r, \theta) = r^3 \sin 3\theta$$

is a function defined on the disk $r < 1$. Find the maximum and minimum values of $u$ on the closed disk. Find all places where these values occur. Justify your answers.

8. (5 points) Suppose that a harmonic function defined on a circle of radius 2 centered at the origin satisfies $u(2, \theta) = 3 + \sin^2(\theta)$ when $r = 2$. Find, with justification, the value of $u$ at the origin.

Initials:

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