# 2.1

Having defined a matrix times a vector, lets turn that idea into a function.

Lets make a function that takes a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and multiplies it by a matrix $A = \begin{bmatrix} 1 & \\ & \end{bmatrix}$

Lets call it

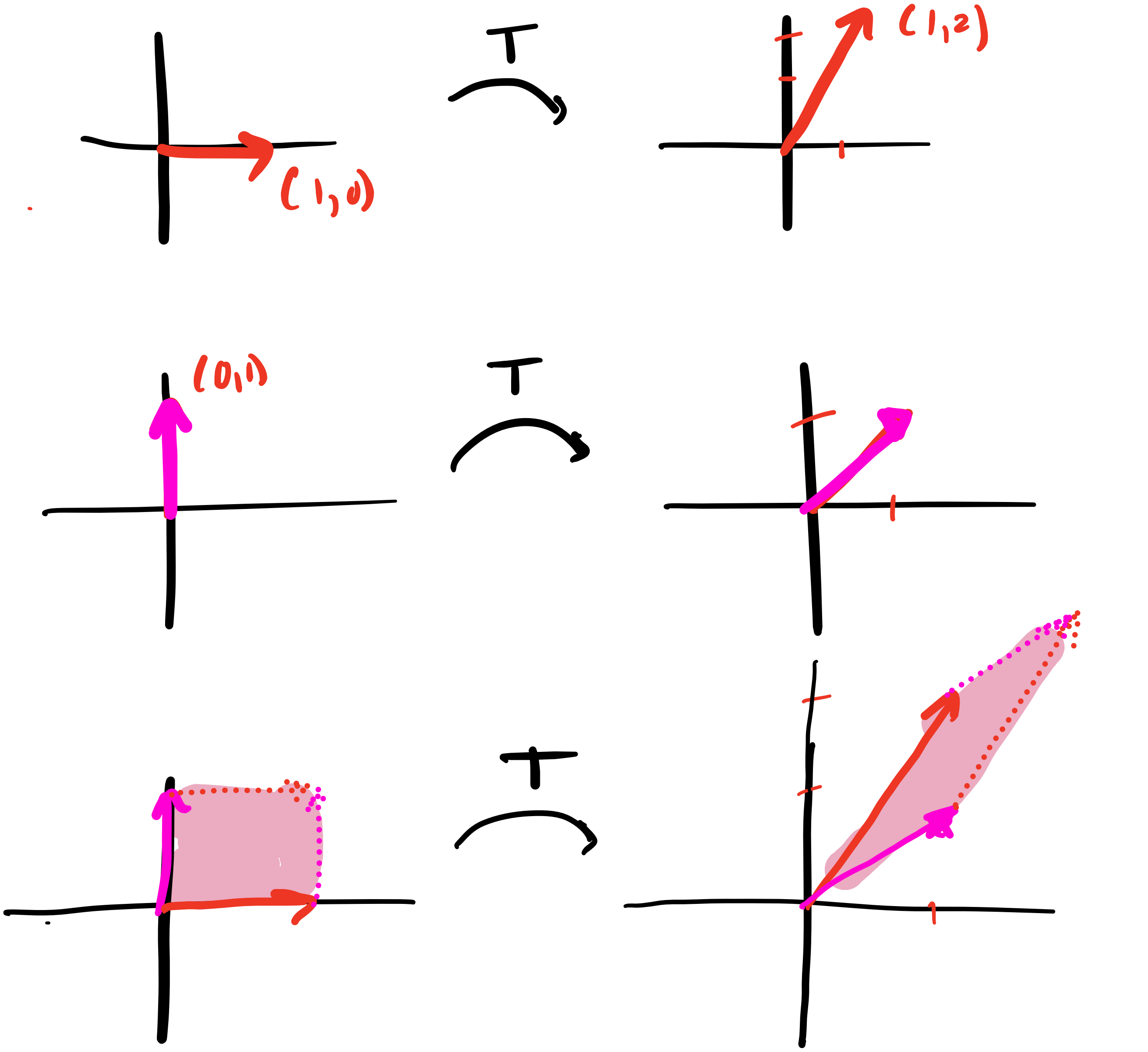
$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

what does $T$ do?

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+y \end{bmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$




<u>Definition:</u> A function $T$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a <u>linear</u> <u>transformation</u> if there exists an $m \times n$ matrix $A$ such that

$$T(\vec{x}) = A\vec{x}.$$

Above, we computed

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$ first column of A

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$ second column of A

The vectors $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$ are called <u>standard</u> <u>basis</u> <u>vectors</u> or <u>elementary</u> vectors.

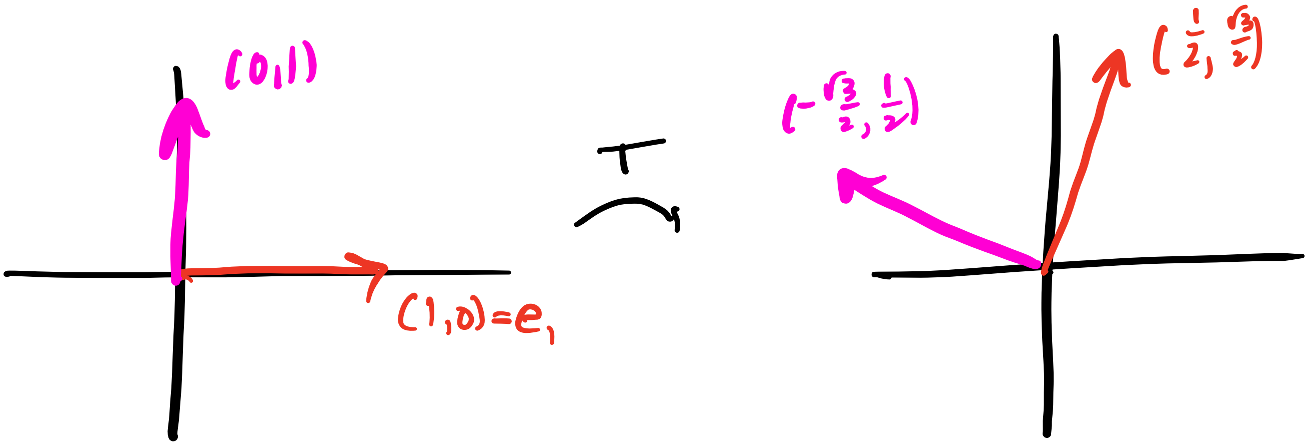
<u>Fact</u>: $A = \begin{bmatrix} | & | & & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_j & \cdots & \vec{v}_n \\ | & | & & & & \end{bmatrix} \vec{e}_j =$

$$= 0\vec{v}_1 + 0\vec{v}_2 + \dots + (1)\vec{v}_j + \dots + 0\vec{v}_n$$

$$= \vec{v}_j$$

We can use this idea to find the matrix of a linear transformation.

Suppose $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates vectors by $\frac{\pi}{3}$ radians




$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix}Te_1 & Te_2\end{bmatrix}\vec{x}$$

$$= \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\vec{x}$$

---

**Theorem:** Properties of linear transformations

---

A transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if

1. $T(\vec{u}+\vec{v}) = T\vec{u} + T\vec{v}$ for all $\vec{u}, \vec{v} \in \mathbb{R}^m$.
2. $T(k\vec{u}) = kT\vec{u}$ for all $k \in \mathbb{R}$, $\vec{u} \in \mathbb{R}^m$.

# Do example 8

An idea from calculus:

Let $T(a+bx+cx^2+dx^3) = \frac{d}{dx}(a+bx+cx^2+dx^3)$
$= b+2cx+3dx^2$

T is linear, since

$$T(p(x)+q(x)) = \frac{d}{dx}(p(x)+q(x))$$
$$= \frac{d}{dx}p(x) + \frac{d}{dx}q(x)$$
$$= T(p(x)) + T(q(x)).$$

and
$$T(kp(x)) = \frac{d}{dx}(kp(x)) = k\frac{d}{dx}p(x)$$
$$= kT(x).$$

How can we write T as a map on vectors?

$$T: \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}.$$

$a+bx+cx^2+dx^3 \longmapsto b+2cx+3dx^2$

$$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$$

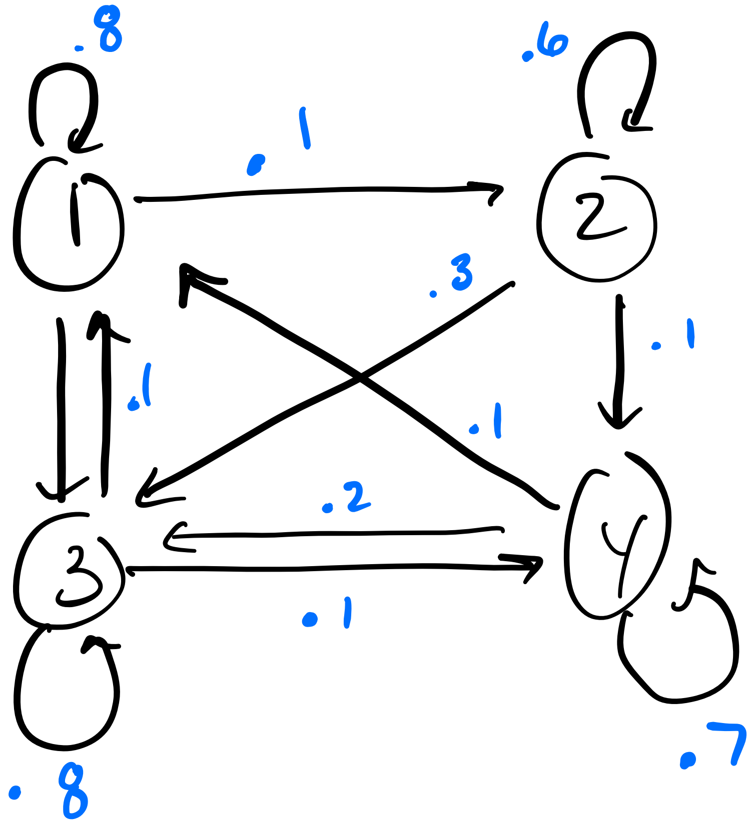
$$T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\3\end{bmatrix}$$

$$\text{so } T\begin{bmatrix}a\\b\\c\\a\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 3\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}a\\2b\\3c\end{bmatrix}.$$

differentiation is linear algebra!
(coupled w/ Taylor series, this idea is very useful).

# Transition matrices

Imagine a network of car rental agencies.




$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \text{initial distribution of cars.}$$

$$y_1 = .8x_1 + \dots + .1x_3 + .1x_4$$

$$y_2 = .1x_1 + .6x_2$$

$$y_3 = .1x_1 + .3x_2 + .8x_3 + .2x_4$$

$$y_4 = \qquad .1x_2 + .1x_3 + .7x_4$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \underbrace{\begin{bmatrix} .8 & 0 & .1 & .1 \\ .1 & .6 & 0 & 0 \\ .1 & .3 & .8 & .2 \\ 0 & .1 & .1 & .7 \end{bmatrix}}_{\text{transition matrix.}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$T(\vec{x}) = \vec{y}$ is linear.

$$T\begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} .8 & 0 & .1 & .1 \\ .1 & .6 & 0 & 0 \\ .1 & .3 & .8 & .2 \\ 0 & .1 & .1 & .7 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} 80 + 10 + 0 \\ 10 + 60 \\ 10 + 30 + 80 + 20 \\ 10 + 10 + 70 \end{bmatrix} = \begin{bmatrix} 100 \\ 70 \\ 140 \\ 90 \end{bmatrix}.$$

in fact, if we continue to apply $T$, we will approach a limit (steady state).

this idea is called a Markov model.

Ex 41: Describe all linear transformations $\mathbb{R}^2 \to \mathbb{R}^1$.

$$\underset{\mathbb{R}^2}{T\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{\mathbb{R}^1}{[c]}$$

$$\underbrace{T\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbb{R}^2} = \underbrace{\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbb{R}^1} = ax+by.$$

as a surface, $z = ax+by$ is a plane through $(0,0,0)$ with normal vector $(a,b,-1)$.

Chapter 2 check in:

what are we doing here?

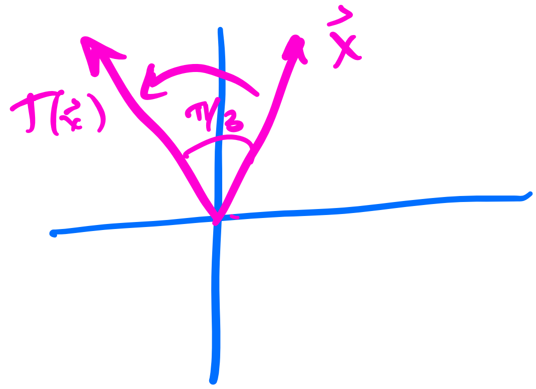
description

"rotates by $\frac{\pi}{3}$"

linear map

$T(\vec{x})$

geometry




rule

$$T(\vec{x}) = A\vec{x}$$

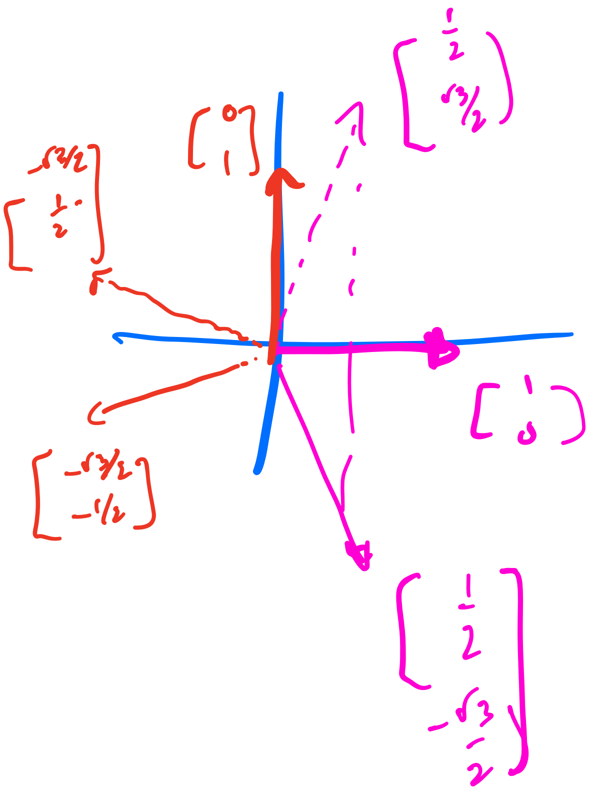
$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \vec{x}$$

A can always be extracted by looking at $T(\vec{e}_i)$

$$A = \left[ T(\vec{e}_1) \quad T(\vec{e}_2) \right] \text{ for } T: \mathbb{R}^2 \to \mathbb{R}^2$$

**Example:**

$T(\vec{x})$ = rotate $\vec{x}$ by $\frac{\pi}{3}$ and then reflect across x-axis




T

$$T(\vec{x}) = A\vec{x}$$

$$= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \vec{x}.$$

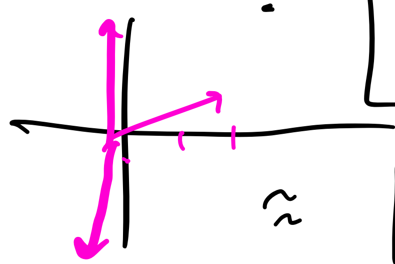
$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Check

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{\sqrt{3}}{2} \\ -\sqrt{3} - \frac{1}{2} \end{bmatrix}$$

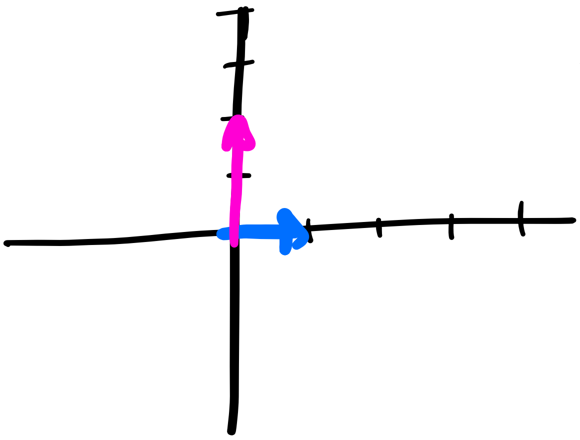


$$\approx \begin{bmatrix} -.2 \\ -1.5 \end{bmatrix}$$

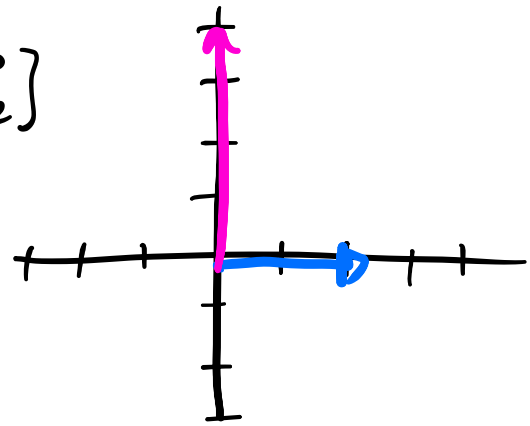
## 2.2 Linear transformations in geometry.

Our goal is to understand linear transformations in general. We'll explore how they can behave by examining 2x2 matrices (which have nice pictures).

a)



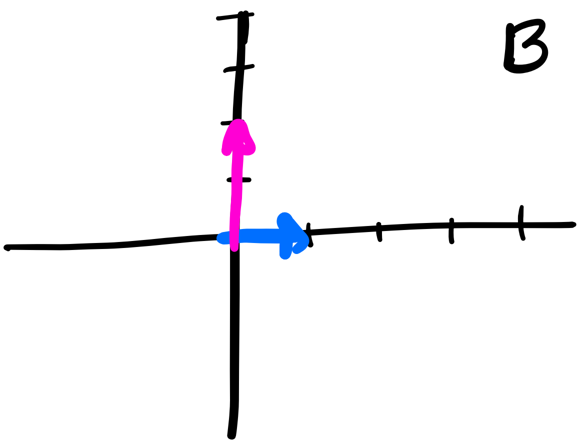
$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$



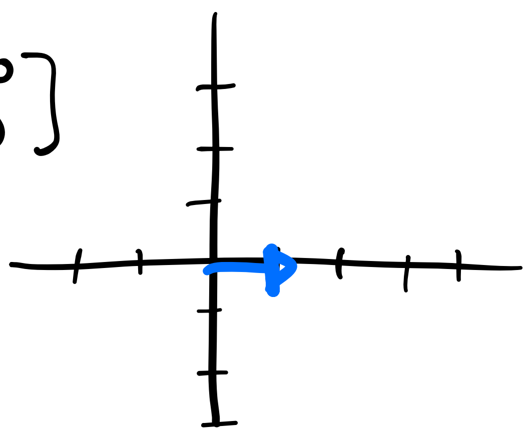
The L doubles in size by a factor of 2.

This is a scaling by 2.

b)

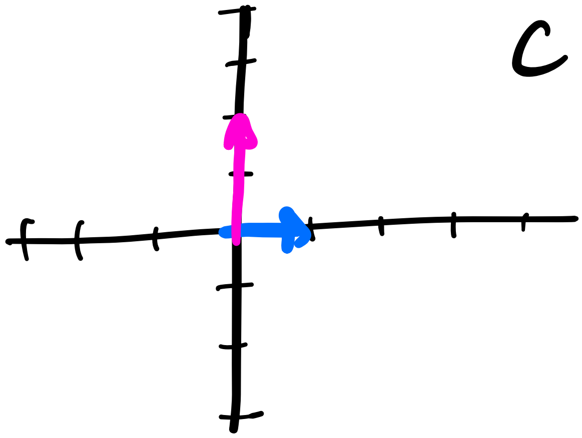


$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

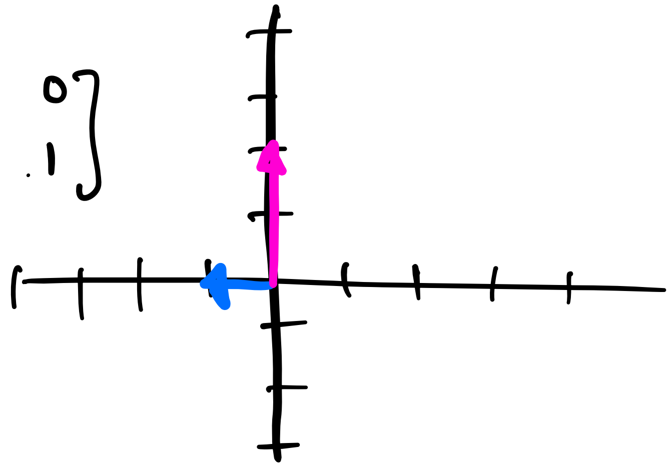


This is an orthogonal projection onto the x-axis.

c)

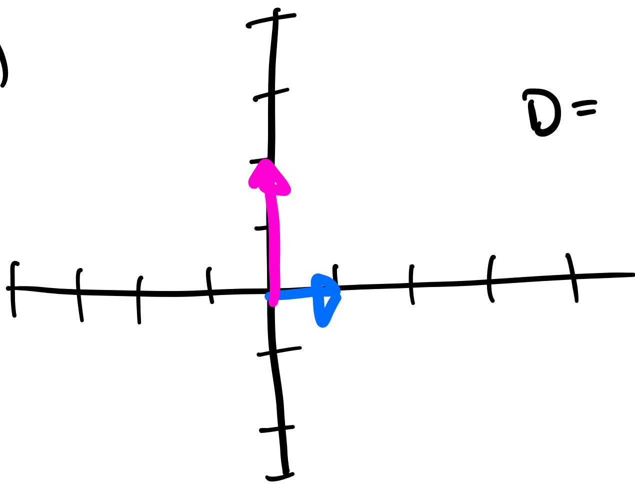


$$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

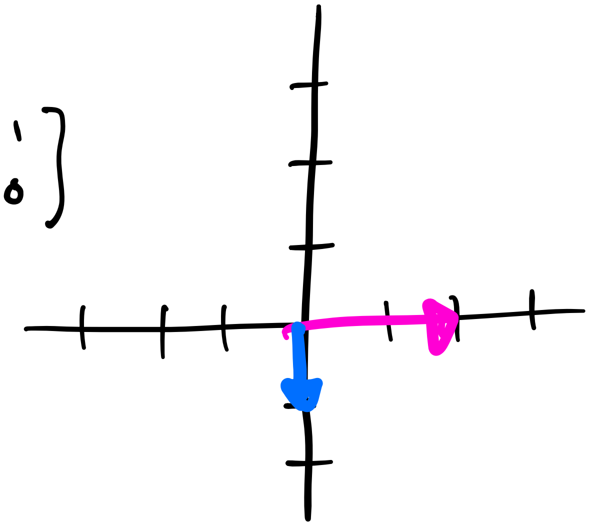


This is <u>reflection</u> about y-axis.

d)

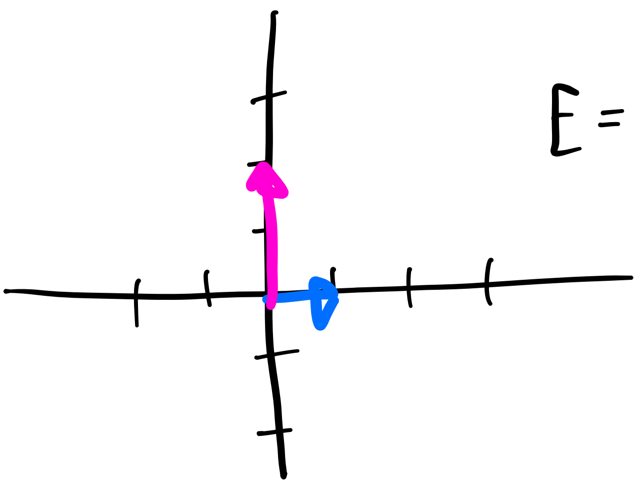


$$D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

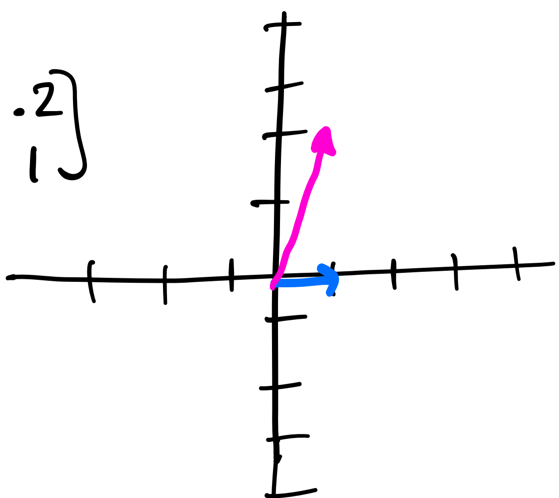


This is a <u>rotation</u> by 270° c.c.w.

e

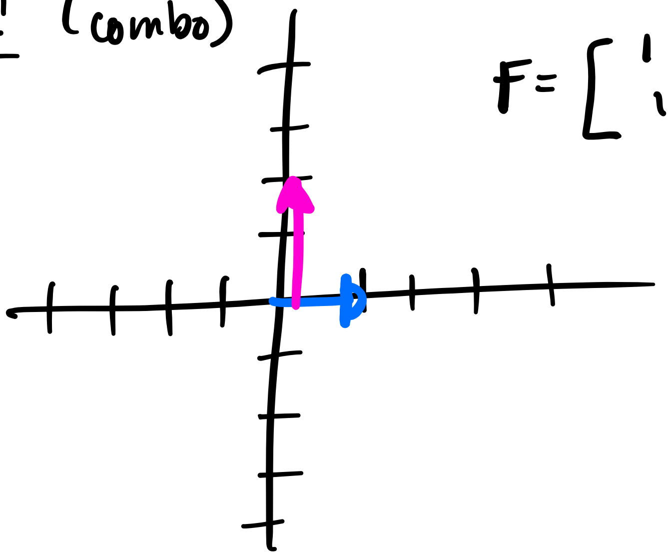


$$E = \begin{bmatrix} 1 & .2 \\ 0 & 1 \end{bmatrix}$$

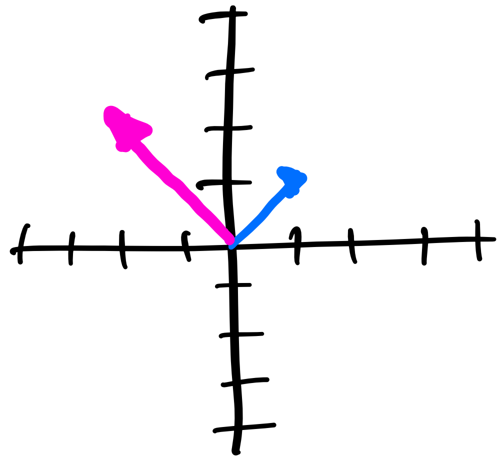


one direction is left alone: this called a <u>sheer</u> transformation.

$\underline{f}$ (combo)



$$F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$



rotation <u>and</u> scaling.

how much rotation? $45°$.

how much scaling? $\sqrt{2}$.

the matrix for a $45°$ rotation should be

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \frac{\sqrt{2}}{2} & \sqrt{2}/2 \end{bmatrix}$$

and scaling

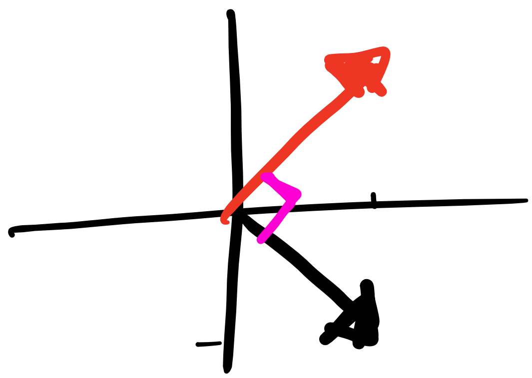
$$\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}.$$

it would be nice to define a type of matrix multiplication so that

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

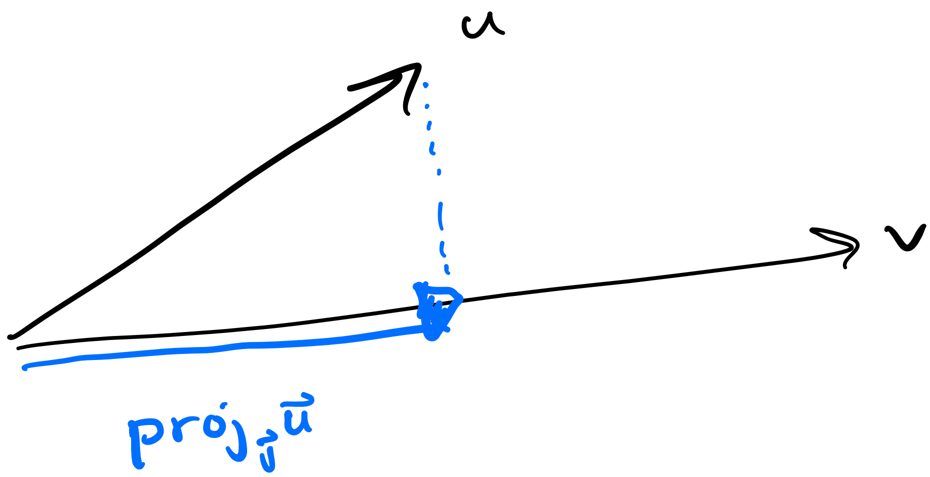
two vectors are <u>orthogonal</u> or <u>perpendicular</u> if $\vec{u} \cdot \vec{v} = 0$.

<u>ex</u>: $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1)(1) + (-1)(1) = 0$

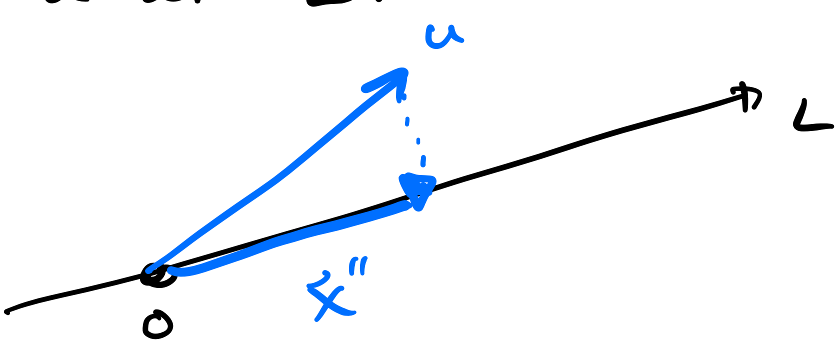


## vector projection

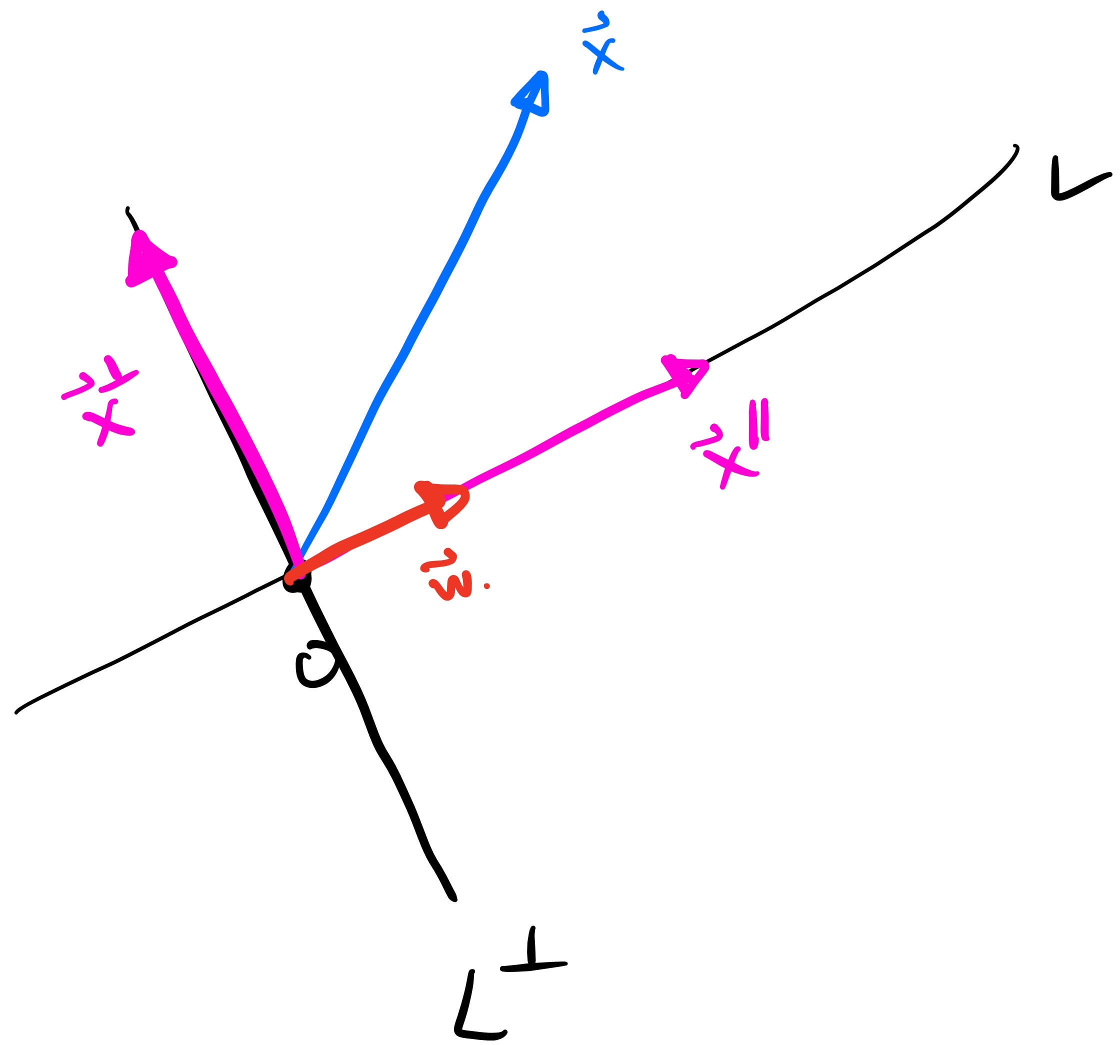
one of the most useful ideas in linear algebra is vector projection.




Bret Shen usually uses $\vec{x}^{\parallel}$, the projection of $\vec{x}$ onto a line $L$.



$\vec{x}^{\parallel} = \text{proj}_L \vec{x}$




$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ ( shows up in the free body diagrams of physical

Seems pretty clear that $\vec{x}^{\parallel} \perp \vec{x}^{\perp}$.

and $(\vec{x} - \vec{x}^{\parallel}) = \vec{x}^{\perp}$

Let $\vec{w}$ be parallel to $L$.

then $\quad \vec{x}^{\parallel} = k\vec{w}$.

$$\vec{w} \cdot \vec{x}^{\perp} = 0$$

so $\vec{w} \cdot (\vec{x} - k\vec{w}) = 0$

so $\vec{w} \cdot \vec{x} - k\, w \cdot w = 0$

so $k = \dfrac{\vec{w} \cdot \vec{x}}{\vec{w} \cdot \vec{w}}$.

so $\vec{x} = \left( \dfrac{\vec{w} \cdot \vec{x}}{\vec{w} \cdot \vec{w}} \right) \vec{w}$.

---

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel} = \left( \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w} \quad \text{for } \vec{w} \parallel L.$$

even better, if we have a unit vector $\vec{u}$ parallel to $L$, $(\vec{u} \cdot \vec{u} = \|\vec{u}\|^2 = 1)$,

then $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\, \vec{u}$ for $\vec{u} \parallel L$.

is $T(\vec{x}) = \text{proj}_L(\vec{x})$ linear? if so, find the

matrix

let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$T(\vec{e}_1) = \text{proj}_L \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= u_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2 \\ u_1 u_2 \end{bmatrix}$$

$$T(\vec{e}_2) = \text{proj}_L \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

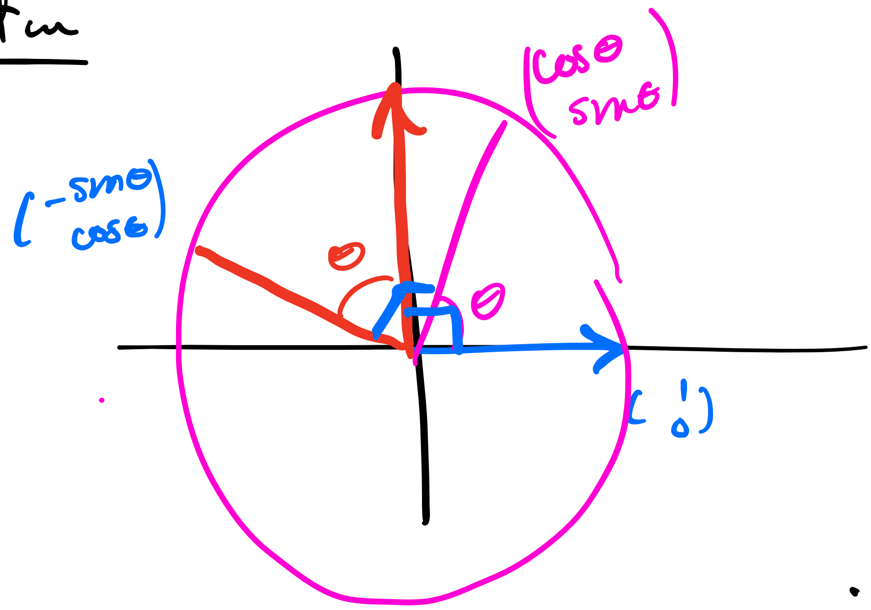
$$= (u_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 u_2 \\ u_2^2 \end{bmatrix}$$

So $T(\vec{x}) = \text{proj}_L(\vec{x}) = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

orthogonal projection is a linear transformation

# Rotation




so $T(\vec{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{x}$.

# 2.3 Matrix Products.

Suppose $T, S$ linear,

$T: \mathbb{R}^n \to \mathbb{R}^m$ $\qquad T(\vec{x}) = B\vec{x}$

$S: \mathbb{R}^p \to \mathbb{R}^n.$ $\qquad S(\vec{x}) = A\vec{x}.$

then $T(S(\vec{x}+\vec{y})) = T(S(\vec{x}) + S(\vec{y}))$

$= T(S(\vec{x})) + T(S(\vec{y}))$

since $T, S$ are linear.

also $T(S(k\vec{x})) = T(k S(\vec{x}))$

$= k T(S(\vec{x}))$

so $T(S(\vec{x}))$ is linear, and must have a matrix.

Let $A = \begin{pmatrix} | & | & & \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_p \\ | & | & & \end{pmatrix}$

then $T(S(\vec{e}_i)) = T(\vec{a}_i) = B\vec{a}_i.$

so $T(S(\vec{x})) = \left( B\vec{a}_1 \; B\vec{a}_2 \cdots B\vec{a}_p \right)\vec{x}.$

So we define matrix multiplication this way.

$$BA = \left( B\vec{a}_1 \quad \cdots \quad B\vec{a}_p \right)$$

note: If $B$ is $n \times m$
and $A$ is $m \times p$
then $BA$ is $n \times p$

---

Example: $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$AB = \left[ \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

properties of matrix multiplication

just like 1 has the property that
$1 \cdot a = a \cdot 1 = a$ for numbers, and so
is called a multiplicative identity.

So is

$$I_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \end{pmatrix}$$

If $A$ is $n \times m$,

$$I_n A = A I_m = A.$$

---

$$(AB)C = A(BC).$$

<u>associativity.</u>

$$A(B+C) = AB + AC$$

$$(B+C)D = BD + CD$$

<u>distributive</u>

generally $BC \neq CB$ (but sometimes it is!)

Ex: $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-2 & -2+2 \\ 3-3 & -2+3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

hmm...

$$BA = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3-2 & 6-6 \\ -1+1 & -2+3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

hmmmmm....

so A and B "cancel" into the identity matrix $I$.

with numbers, if $ab=1$,

then $b=a^{-1}$.

with $n \times n$ matrices, if

$AB=I$, then $B=A^{-1}$.

lets chase then idea.

---

## 2.4 Invertible Matrices and transformations

A function $T: X \to Y$ is called invertible if $T(x)=y$ has a unique solution $x$ in $X$ for any $y$ in $Y$.

Ex: if $T(x)=3x$ then $T(x)=y$ has solution

$3x=y$

$x=\frac{1}{3}y$

for all $y$,

so $T(x)=3x$ is invertible with inverse $T(x)=\frac{1}{3}x$.

In this case,

$$T^{-1}(y)=x \quad \text{means} \quad T(x)=y$$

so $T(T^{-1}(y)) = y$

$T^{-1}(T(x)) = x.$

---

if some function $L: Y \to X$ has

$L(T(x))=x$ and $T(L(y))=y$,

then $L=T^{-1}$.

---

$$(T^{-1})^{-1}=T.$$

---

A square matrix $A$ is <u>invertible</u>

if $T(x)=Ax$ is invertible.

If this holds, then $A^{-1}$ is the matrix so that $T^{-1}(\vec{y}) = A^{-1}\vec{y}$.

A $n \times n$ matrix $A$ is invertible iff

$\mathrm{rref}(A) = I_n$.

is $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ invertible?

$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{matrix} \\ -I \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{matrix} -2II \\ \\ \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yes.

How can we find $A^{-1}$? ... ... row reduction.

if $\mathrm{rref}[\,A \mid I\,] = [\,I \mid B\,]$ then $B = A^{-1}$.

if not, $A$ is not invertible.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \begin{matrix} \\ -I \end{matrix}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \begin{matrix} -2II \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\sim [\; I \;|\; A^{-1} ].$$

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$$A^{-1}A = AA^{-1} = I$$

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what is the inverse of $AB$?

$$(AB)C = I$$

$$\underbrace{A^{-1}A}BC = A^{-1}I$$

$$IBC = A^{-1}$$

$$BC = A^{-1}$$

$$B^{-1}BC = B^{-1}A^{-1}$$

$$IC = B^{-1}A^{-1}$$

$$C = B^{-1}A^{-1}$$

so $(AB)^{-1} = B^{-1}A^{-1}$.

the <u>determinant</u> of a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

is $\det(A) = ad - bc$.

why?

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= ad-bc \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if $ad-bc \neq 0$,

$$\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{I}$$

so $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$\det\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = 1(3) - 2(1) = 3-2 = 1 \neq 0.$

so $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ is invertible.

ad $\quad \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}.$$

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Invertible Matrix theorem:

Let $A$ be $n \times n$ TFAE.

1. $A$ is invertible.
2. $\det(A) \neq 0$
3. there is a matrix $B$ so $BA = AB = I$.
4. $A\vec{x} = 0$ has exactly one solution $\vec{x} = 0$.
5. $A\vec{x} = b$ has exactly one solution for any $\vec{b}$.
6. $\text{rref}(A) = I_n$
7. $\text{rank}(A) = n$.