Definitions are key!

Given a system of linear equations, the goal is to get it in reduced row echelon form and then solve it.

Definition: RREF

An augmented matrix or linear system is in reduced row echelon form if

1. the first non-zero entry of each row is a 1, which is called the leading entry.
2. if a row contains a leading 1, every other entry above and below that 1 is a 0.
3. if a row contains a leading 1, the row above it has a leading 1 to the left.

Example:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Elementary Row operations.

1. divide a row by a non-zero scalar
2. subtract a multiple of a row from another row
3. swap two rows.

Generally, we want to work with augmented matrices but keep the system it represents in mind when necessary.

Row reduction is our <u>primary</u> <u>tool</u>. Almost everything we learn will involve row reducing matrices as a key step.

$\boxed{1.3}$ Consider the systems represented by

(1) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

System (1) is
$$x_1 = 1$$
$$x_2 = 1.$$

it has one, <u>unique</u> solution

System 2 is $x_1 + x_2 = 1$

or $x_1 = 1 - x_2$.

x_2 can be chosen freely, which then determines x_1.

If $x_2 = t$,

$x_1 = 1 - t$.

System 2 has <u>infinitely many</u> solutions.

System 3 is $x_1 = 1$

$0 = 1$.

This is impossible, and hence the system has <u>no</u> solutions.

We call this an <u>inconsistent</u> system.

The number of solutions possible for a given system is related to the number of leading 1s when the system is row reduced.

Definition: RANK

The **rank** of a matrix is the number of leading 1s in $\mathrm{rref}(A)$.

$$A \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{rref}(A)$$

$\mathrm{rank}(A) = 2.$

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consider

$$\begin{array}{cc} & n \\ m & \left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \end{array}$$

That is, a system of m linear equations in n variables.

① rank $A \le n$, rank $A \le m$.

② If system is inconsistent,
rank $A < m$.

③ If exactly one solution,
rank $A = m$.

④ If infinite solutions,
rank $A < n$.

rank $A = m$
$\Rightarrow$ consistent

rank $A \neq m$
$\Rightarrow$ ∞ or no solution.

rank $A = n$
$\Rightarrow$ ∞ or one solution

Important Example:

If A is $n \times n$,
then $[A \,\vdots\, \vec{b}]$ has a unique solution
if and only if $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ \vdots & & & \ddots \\ & & & & 1 \end{pmatrix}$

if and only if $\text{rank}(A) = n$.

this is called the <u>identity</u> matrix I_n.

Matrix Algebra

In many ways, matrices and like vectors.

Suppose A, B are both $m \times n$. k a scalar.

$$A + B = \begin{bmatrix} a_{11} & \cdots \\ \vdots & \ddots \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots \\ \vdots & \ddots \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$kA = k \begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots \\ \vdots & & \ddots \end{bmatrix}.$$

Just like vectors, there are ways to multiply matrices.

dot product

Suppose $\vec{v} = (v_1, \ldots, v_n)$ $\vec{w} = (w_1 \ldots w_n)$.

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \ldots + v_n w_n.$$

Matrix multiplied by vector.

Suppose A is $m \times n$ and $\vec{x} = (x_1 \ldots x_n)$

then $A\vec{x} = \begin{bmatrix} \vec{w}_1 \\ \vdots \\ \vec{w}_m \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot x \\ \vec{w}_2 \cdot x \\ \vdots \\ \vec{w}_m \cdot x \end{bmatrix}$

Examples:

- $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ 6 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 10 & 13 & 16 \end{bmatrix}$$

- $$3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} = (1)(2) + (2)(6) + (3)(10)$$
$$= 2 + 12 + 30$$
$$= 44$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (1,2,3) \cdot (1,2,1) \\ (4,5,6) \cdot (1,2,1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 4 + 3 \\ 4 + 10 + 6 \end{bmatrix}$$

OR $$= \begin{bmatrix} 8 \\ 20 \end{bmatrix}$$

$$(1) \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (2) \begin{bmatrix} 2 \\ 5 \end{bmatrix} + (1) \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 10 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \end{bmatrix}.$$

Column Version

Suppose A is $m \times n$ and $\vec{x} = (x_1 \dots x_n)$

$$A\vec{x} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \vec{x} = \underbrace{x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n}_{\text{special type of sum.}}$$

Linear Combination

A linear combination of vectors $\vec{v}_1 \dots \vec{v}_n$ is

a sum $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ for scalars

$c_1 \dots c_n$.

Question: Can $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ be a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$?

That is, can we find c_1, c_2 s.t.

$$\underbrace{c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}_{} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ matrix form.

$$\begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$$

$c_1 = -1/3$

$c_2 = 1/3$.

<u>So</u>

$$\left(-\tfrac{1}{3}\right)\begin{pmatrix}1\\2\\3\end{pmatrix} + \left(\tfrac{1}{3}\right)\begin{pmatrix}4\\5\\6\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\2 & 5\\3 & 6\end{pmatrix}\begin{pmatrix}-1/3\\1/3\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

<u>Def: Matrix form</u>

The matrix form of a linear system is

$A\vec{x} = \vec{b}$ where A is the coefficient matrix, and $\vec{x}$ is a vector of unknowns.

This is our first big picture moment.

$ax = b$ we would "just" divide by a.

$A\vec{x} = \vec{b}$ ... can we somehow "divide" by a matrix?