## Finding Eigenvectors.

Once we know an eigenvalue, we can solve for eigenvectors using row reduction! (imagine that)

**Def**: Eigenspace

Suppose $\lambda$ is an eigenvalue of $A$. The kernel of $A-\lambda I$ is called the eigenspace associated with $\lambda$, denoted $E_\lambda$.

$$E_\lambda = \ker(A-\lambda I) = \{\vec{v} : A\vec{v} = \lambda\vec{v}\}.$$

Example: diagonalize $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ if possible.

$\boxed{\text{Step 1:}}$ Find eigenvalues.

$$0 = \det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}$$

$$= (1-\lambda)(3-\lambda) - 8$$

$$= \lambda^2 - 4\lambda + 3 - 8$$

$$= \lambda^2 - 4\lambda - 5$$

$= (\lambda - 5)(\lambda + 1)$

So $\lambda = 5$ $\quad \lambda = -1$.

$\boxed{\text{STEP 2:}}$

$\underline{\lambda = 5}$

$E_5 = \ker(A - 5I)$

$= \ker\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}$

$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x = \begin{pmatrix} 1/2\,\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

we could also choose $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$E_5 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

$E_{-1} = \ker\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$

$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\vec{x} = \alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

$$A\begin{pmatrix}1\\2\end{pmatrix} = 5\begin{pmatrix}1\\2\end{pmatrix}$$
$$A\begin{pmatrix}-1\\1\end{pmatrix} = -1\begin{pmatrix}-1\\1\end{pmatrix}$$

so $S = \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}$

$B = \begin{pmatrix}5 & 0\\0 & -1\end{pmatrix}$

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diagonalize $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ if possible.

$\boxed{1}$

$$0 = \det\begin{pmatrix}1-\lambda & 1\\0 & 1-\lambda\end{pmatrix} = (1-\lambda)^2$$

so $\lambda = 1$ alg multiplicity 2.

$\boxed{2}$

$$E_1 = \ker\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$

$$\left(\begin{array}{cc|c}0 & 1 & 0\\0 & 0 & 0\end{array}\right)$$

$x_1 =$ free

$x_2 = 0$

$$\vec{x} = \begin{pmatrix}\alpha\\0\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}.$$

there aren't enough linearly independent eigenvectors!

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \vec{v}_2 \overset{?}{=} \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

not a basis!

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so $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ cannot be diagonalized

A matrix that cannot be diagonalized is called defective.

**Def** The geometric multiplicity of an eigenvalue $\lambda$ is the dimension of $E_\lambda$.

in our previous example,

$\text{almu}(1) = 2$

$\text{gemu}(1) = 1$

$\text{almu}(1) \neq \text{gemu}(1)$

so $A$ is not diagonalizable.

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$\text{geomu}(\lambda) \leq \text{algmu}(\lambda)$.

If every eigenvalue has alg mult 1,
then every alg. mult($\lambda$) = ge. mult($\lambda$) = 1

and so

$$\boxed{\text{if } A \text{ has } n \text{ distinct eigenvalues, } A \text{ is diagonalizable.}}$$

More generally,
if every eigenvalue $\lambda$ of $A$ has
almu($\lambda$) = gemu($\lambda$), then $A$ is
diagonalizable.

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Show that every matrix of the form

$$\begin{pmatrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{pmatrix}$$ cannot be diagonalized

$\det(A-\lambda I) = (c-\lambda)^3$ so $\lambda = c$ is
an eigenvalue,
almu($c$) = 3.

$$\ker(A - cI) = \ker\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1$ free

$x_2 = 0$

$x_3 = 0$

$$\vec{x} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$\operatorname{gemu}(c) = 1$!!.

$\operatorname{almu}(c) \neq \operatorname{gemu}(c)$

so $\begin{pmatrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{pmatrix}$ cannot be diagonalized.