Finding eigenvalues.

λ is an eigenvalue of A if for some vector $\vec{v}$,

$$A\vec{v} = \lambda\vec{v}.$$

lets find a condition that reduces the number of unknowns

$$A\vec{v} - \lambda\vec{v} = 0$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

If such a vector existed, $\ker(A-\lambda I) \neq \{0\}$, and $A-\lambda I$ wouldn't be invertible... ideas?

$$\det(A - \lambda I) = 0 \Rightarrow (A-\lambda I)\vec{v} = 0$$

$$\Rightarrow A\vec{v} = \lambda\vec{v}.$$

Thm: A $n\times n$, λ a scalar.

λ is an eigenvalue of A if and only if

$$\boxed{\det(A-\lambda I)=0}$$

Characteristic equation

$f_A(\lambda) = \det(A-\lambda I)$ is the characteristic polynomial.

Ex: Find the eigenvalues of

$$A=\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}.$$

$$0=\det(A-\lambda I)$$

$$=\det\left(\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$=\det\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}$$

$$=(2-\lambda)(1-\lambda)-6$$

$$=\lambda^2-3\lambda+2-6$$

$$=\lambda^2-3\lambda-4.$$

$$f_A(\lambda)=\lambda^2-3\lambda-4$$

$$=(\lambda-4)(\lambda+1)$$

So $\lambda - 4 = 0$ or $\lambda + 1 = 0$

$\lambda = 4$ $\qquad$ $\lambda = -1.$

already, we know that

$$B = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

but to finish diagonalizing, we would need $\vec{v}_1, \vec{v}_2$ eigenbasis.

Example: Find eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$f_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)(1-\lambda)^3$$

roots are $\lambda = 2$ multiplicity 1

$\lambda = 1$ multiplicity 3

Def: an eigenvalue λ_0 of A has algebraic multiplicity k if λ_0 is a root of order k of $f_A(\lambda)$.

$$f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda) \quad \text{w/} \quad g(\lambda_0) \neq 0.$$

$$\text{almu}(\lambda_0) = k.$$

Example:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$f_A(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix}$$

$$= \lambda^2 (3 - \lambda) \qquad \text{verify}$$

eigenvalues are $\underbrace{0, 0}_{\text{mult } 2}, \underbrace{3}_{\text{mult } 1}$

Future Case:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$f_A(x) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

$$= \lambda^2 + 1$$

$\lambda^2 + 1 = 0$ has no real solutions

(in fact $\lambda = \pm i$ are the eigenvalues. we'll see this in 7.5)