# Diagonalization

An $n \times n$ matrix is diagonal if the only non-zero entries appear on the main diagonal.

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

Diagonal matrices are great.

Suppose

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$A^5$, $B^5$, rank $A$, rank $B$, det $A$, det $B$

all way easier to use $A$.

when possible, we prefer to work with diagonal matrices.

**Def**: A transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is diagonalizable if the matrix of $T$ with respect to some matrix is diagonal.

when can this happen?

Recall $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ and $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A\vec{v}_1 = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4\vec{v}_1$$

$$A\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\vec{v}_2.$$

so $B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ with basis $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

More generally, if there is a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $A$ and numbers $\lambda_1, \dots \lambda_n$ so that

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$
$$\vdots$$
$$A\vec{v}_n = \lambda_n \vec{v}_n$$

then $B = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & \\ 0 & \lambda_2 & 0 & \cdots & \\ 0 & 0 & \lambda_3 & \cdots & \\ \vdots & & & & \vdots \\ & & & \cdots & \lambda_n \end{pmatrix}$

So the big question is, how can we find these vectors $\vec{v}_1, \dots \vec{v}_n$?

<u>Ex:</u> $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ $\quad v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$A\vec{v}_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} = \boxed{2}\vec{v}_1.$$

$$A\vec{v}_2 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boxed{1}\vec{v}_2$$

$$A\vec{v}_3 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \boxed{1}\vec{v}_3.$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are a basis for $\mathbb{R}^3$.

so $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2, 1, 1 are called <u>eigenvalues</u>

and

$\vec{v}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are called

<u>eigenvectors</u>

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ is called an eigenbasis as it is a basis for $\mathbb{R}^3$.

A matrix $A$ is diagonalizable if there exists one eigenbasis for $A$.
If it does, with $A\vec{v}_1 = \lambda_1 \vec{v}_1, \ldots, A\vec{v}_n = \lambda \vec{v}_n$,

then
$$S = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} \quad B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

diagonalize $A$, meaning
$$S^{-1}AS = B.$$

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Invertible matrix theorem, final form.
The following are equivalent for an $n \times n$ matrix $A$.

1. $A$ is invertible
2. $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in $\mathbb{R}^n$
3. $\text{rref}(A) = I_n$.
4. $\text{rank}(A) = n$
5. $\text{im}(A) = \mathbb{R}^n$
6. $\ker A = \{0\}$.

7. $A\vec{x}=0$ has only the solution $\vec{x}=0$.

8. the columns of $A$ are a basis for $\mathbb{R}^n$.

9. the columns of $A$ span $\mathbb{R}^n$

10. the columns of $A$ are linearly independent.

11. $\det A \neq 0$.

12. $0$ is <u>not</u> an eigenvalue of $A$.