Determinants of larger matrices

(this is very different from the book's presentation)

Having established a formula for the determinant of a 2×2 or 3×3 matrix, we'd like some way to extend to 4×4 and more generally $n\times n$.

Unfortunately, Sarrus's formula fails. Another idea is needed.

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

$$\begin{aligned} \det A = {} & a_{11}(a_{22}a_{33} - a_{32}a_{23}) \\ & + a_{21}(a_{32}a_{13} - a_{12}a_{33}) \\ & + a_{31}(a_{12}a_{23} - a_{12}a_{23}) \end{aligned}$$

$$\begin{aligned} & a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \\ & - a_{21} \det\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} \\ & + a_{31} \det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} \end{aligned}$$

Def: Let A_{ij} be the matrix resulting from eliminating row i and column j from A.

A <u>minor</u> of A is $\det A_{ij}$.

Notation:

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + a_{31}\det(A_{31})$$

we can do this along any row or column, following the sign matrix

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For example, $\det A = -a_{12}\det(A_{12})$
$+ a_{22}\det(A_{22})$
$- a_{32}\det(A_{32})$

Example:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\det(A) = a_{21}\det(A_{21}) + a_{22}\det(A_{22}) - a_{23}\det(A_{23})$$

$$= -3(0) + (-1)(0) - 2(1) = -2.$$

<u>or</u>

$$\det(A) = a_{31}\det(A_{31}) - a_{32}\det(A_{32}) + a_{33}\det(A_{33})$$

$$A_{31} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \quad A_{32} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \quad A_{33} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

$$\det(A) = 0(2) - 1(2) + 0(-1) = \boxed{-2}$$

This works for larger matrices:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\det A = -\cancel{a_{21}\det \overset{0}{A_{21}}} + \cancel{a_{22}\det \overset{0}{A_{22}}} - a_{23}\det A_{23} + a_{24}\det A_{24}$$

$$= -3\det\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} = 1(6-4) - 2(3-4) + 1(1-2)$$

$$= 2 + 2 - 1 = 3.$$

$$\det\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{pmatrix} = 1(-2-3) - 2(-1-3) \cancel{+0(}$$

$$= 3$$

$= -3(3) + 1(3) = -6$

(A is invertible.)

Computational fact: A triangular matrix has determinant equal to the product of diagonal entries.

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = (1)(2) = 2$$

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} = (1)(0)(3) = 0.$$

If S is invertible, $\det(S^{-1}) = \frac{1}{\det(S)}$.

ex: Find $\det S^{-1}$ for $S = \begin{pmatrix} 1 & 6 \\ 10 & 3 \end{pmatrix}$

$$\det(S) = 3 - 60 = -57$$

so $\det(S^{-1}) = \frac{-1}{57}$.