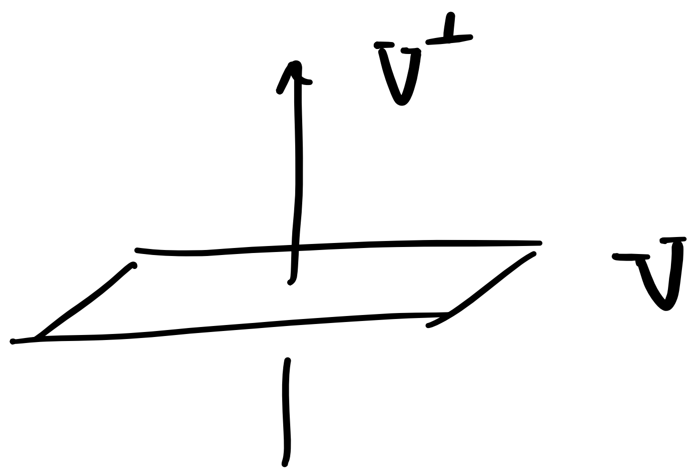
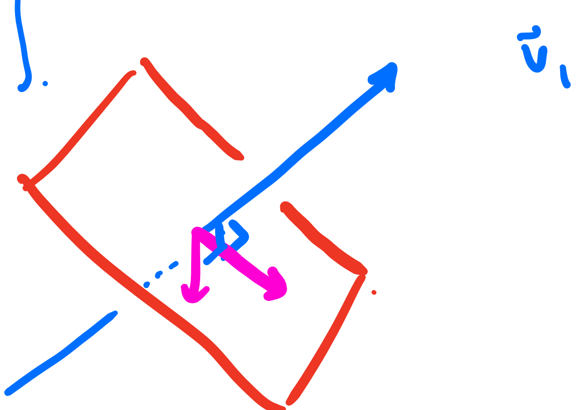
Orthogonal Complements and Least Squares



<u>Def</u>:

Given a subspace V of $\mathbb{R}^n$, the <u>orthogonal complement</u> of V, denoted $V^\perp$, is the space of vectors that are orthogonal to V.

<u>Example</u>: Let V be the line spanned by $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.



$V^\perp$ should be a plane. How can we find it?

$\begin{bmatrix}1\\2\\3\end{bmatrix}$ is the normal vector to $x+2y+3z=0$

so $V^\perp$ is the plane $x+2y+3z=0$

Not linear algebraic and hard to see how this example generalizes.

Let's continue to write $V^\perp$ as a span.

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\end{array}\right]$$

y free $=\alpha$
z free $=\beta$

$x+2\alpha+3\beta=0$

$x=-2\alpha-3\beta$

$$\begin{bmatrix}-2\alpha-3\beta\\ \alpha\\ \beta\end{bmatrix}=\alpha\begin{bmatrix}-2\\1\\0\end{bmatrix}+\beta\begin{bmatrix}-3\\0\\1\end{bmatrix}$$

$$=\operatorname{span}\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}$$

so $V^\perp=\operatorname{span}\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}$

$=\ker\{\begin{bmatrix}1 & 2 & 3\end{bmatrix}\}.$

Def: the transpose of a vector or a matrix change its rows into columns and columns into rows. We write A^T or $\vec{v}^T$ for a transpose.

for a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$\vec{v}^T = [v_1 \cdots v_n]$$

for a matrix $A = [\vec{v}_1 \cdots \vec{v}_n]$

$$A^T = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix}$$

columns become rows.

---

that is, $V^\perp = \ker\{\vec{v}_1'\}$ in our previous example.

Thm: $(\text{im} A)^\perp = \ker(A^T)$

Example: let $V = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix} \right\}$.

$V = \text{im} A$ where $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$

$V^\perp = (\text{im} A)^\perp = \ker A^T$

$A^T = \begin{bmatrix} 1 & -2 & 2 & 0 \\ 1 & -1 & 3 & 1 \end{bmatrix}$

$\ker A^T$ is solutions to $A^T x = 0$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 0 & 0 \\ 1 & -1 & 3 & 1 & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$x_3 = \alpha \qquad x_1 = -4\alpha - 2\beta$

$x_4 = \beta \qquad x_2 = -\alpha - \beta$

$$\vec{x} = \begin{bmatrix} -4\alpha - 2\beta \\ -\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

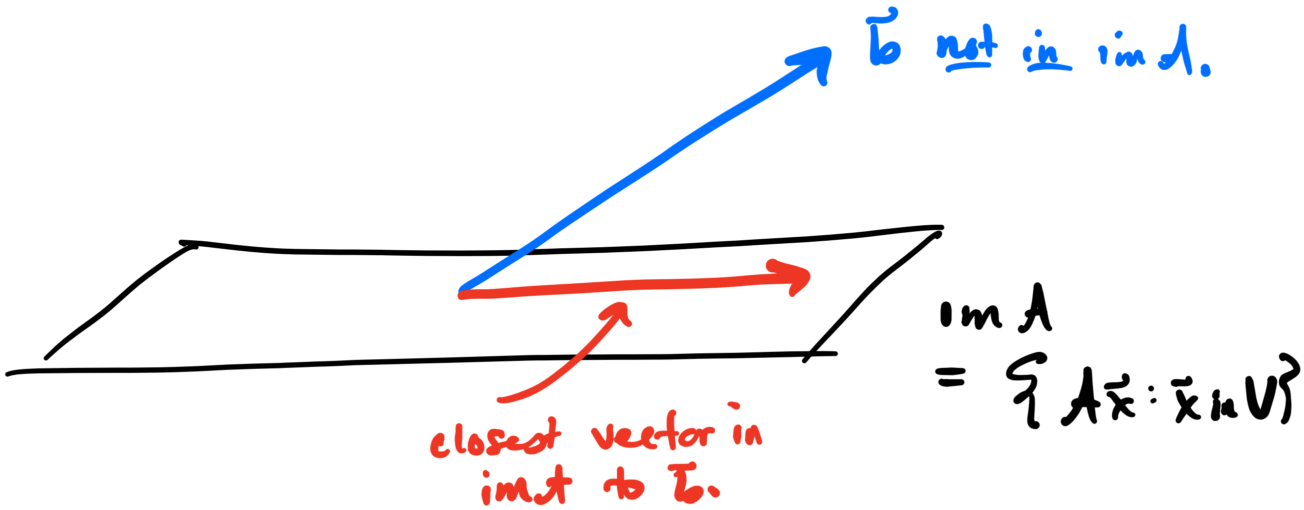
$$= \operatorname{span}\left\{ \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= V^{\perp}$$

Least Squares

Consider an inconsistent problem

$A\vec{x} = \vec{b}$. (That is, there is <u>no</u> solution.)



So no solution, hence inconsistent.

how can we get as close as possible?

project $\vec{b}$ onto im A and solve for $\vec{x}$!

process: $A\vec{x} = \vec{b}$ is inconsistent.

project $\vec{b}$ onto im A.

$\text{proj}_{\text{im}A}\,\vec{b} = A\vec{x}^*$ for some $\vec{x}^*$.

find $\vec{x}^*$, which is called the <u>least</u> <u>squares</u> <u>solution</u>.

we can use the transpose to compute this.

Thm: The least squares solutions to

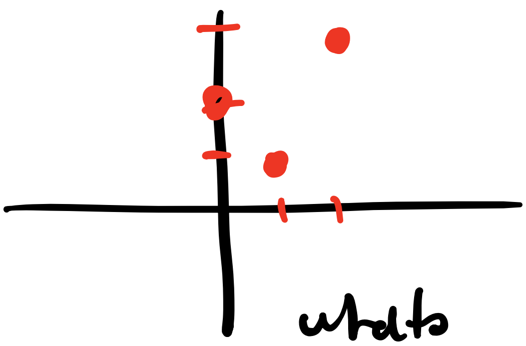
$$A\vec{x} = \vec{b} \quad \text{are the solutions}$$

to $$A^T A \vec{x}^* = A^T \vec{b}$$

Corollary: If $\ker(A) = \{0\}$ (lin ind cols),

then $$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

Example: Find a line through $(1,1)$, $(2,3)$, $(0,2)$



obviously can't be done.

what's the best we can do?

$$y = mx + b$$

$$1 = 1m + b$$
$$3 = 2m + b$$
$$2 = 0m + b$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}_{\vec{b}}$$

inconsistent: multiply through by A^T.

$$\underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\vec{x}^+} = \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}_{\vec{b}}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

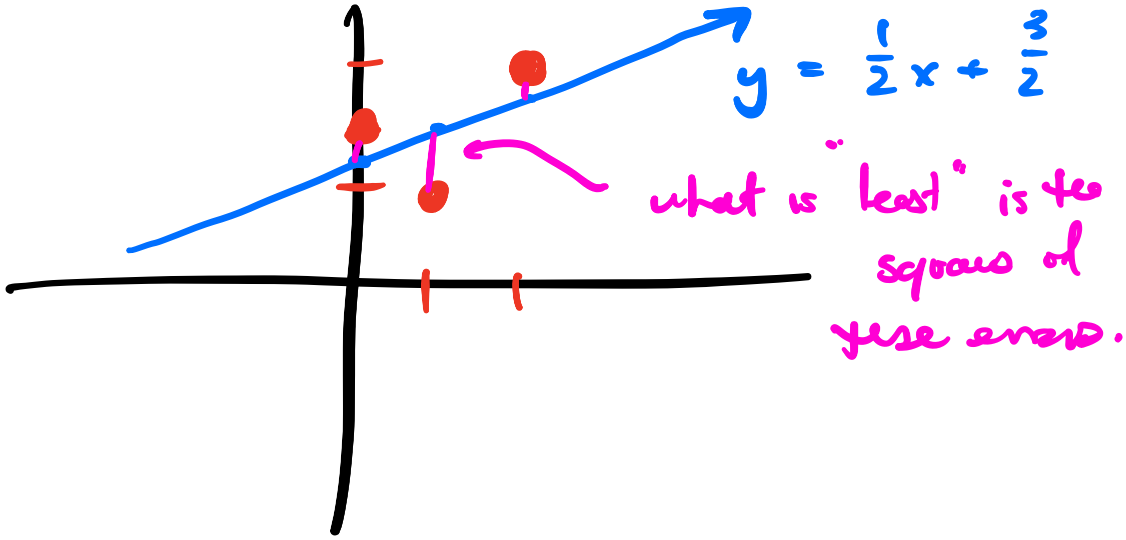
$$\left[\begin{array}{cc|c} 5 & 3 & 7 \\ 3 & 3 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 5 & 3 & 7 \end{array}\right]$$

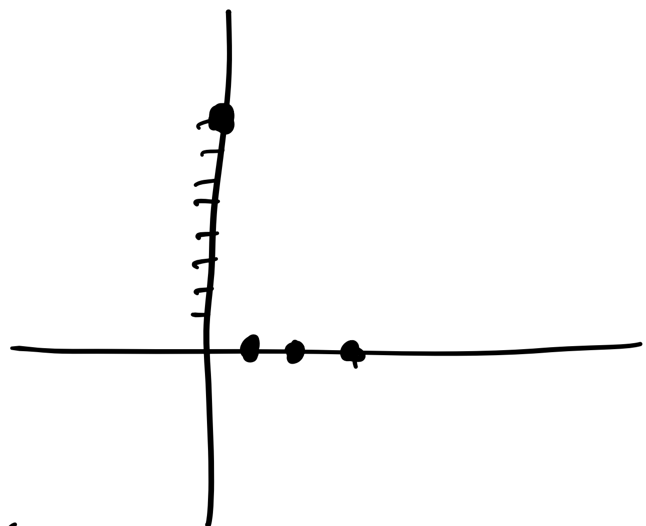
$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 3/2 \end{array}\right] \sim \begin{array}{ccc} m & b & \\ \hline \end{array}\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \end{bmatrix}$$

$$m = \frac{1}{2}$$
$$b = 3/2$$



Fit a quadratic function to $(0,27)$, $(1,0)$, $(2,0)$, $(3,0)$



$p(x) = ax^2 + bx + c$

$27 = 0a + 0b + c$

$0 = a + b + c$

$0 = 4a + 2b + c$

$0 = 9a + 3b + c$

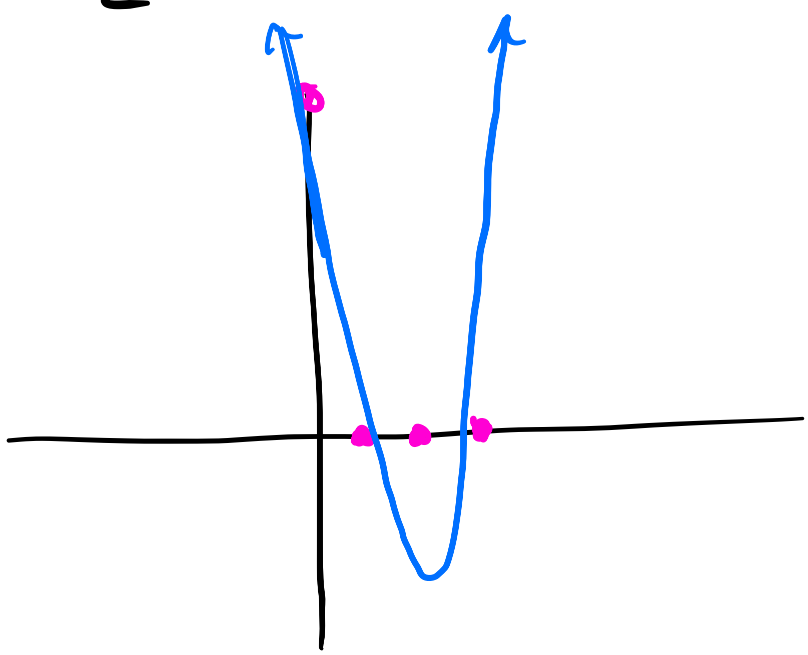
$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 27 \\ 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 0 \\ 9 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

inconsistent.

$$\begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 98 & 36 & 14 \\ 36 & 14 & 6 \\ 14 & 6 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 27 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 27/4 \\ 0 & 1 & 0 & -567/20 \\ 0 & 0 & 1 & 513/20 \end{array}\right]$$



$$y = 6.75x^2 - 28.35x + 25.65$$