5.2 Gram-Schmidt

suppose $\bar{V} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} \right\}$.

I'd love to compute

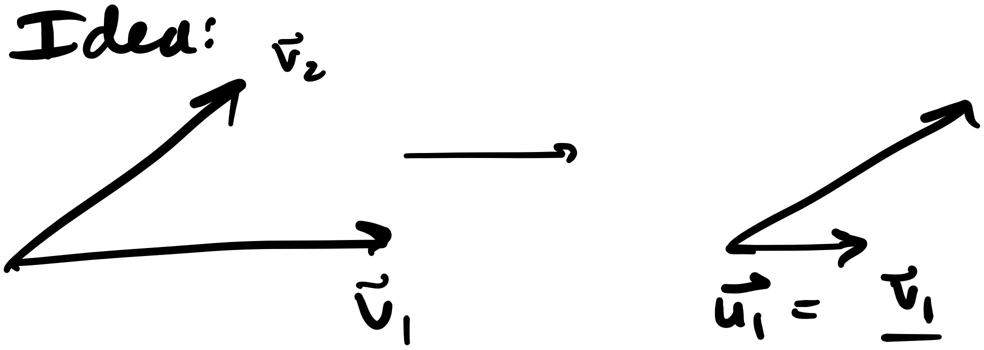
$\text{proj}_{\bar{V}} \vec{x}$ but I can't! $\left(= (\vec{u}_1 \cdot \vec{x}_1)\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 \right)$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 9 \\ 9 \\ 1 \end{bmatrix} = 20 \neq 0$$

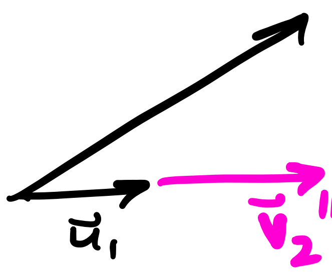
<u>not</u> orthogonal.

what can I do?

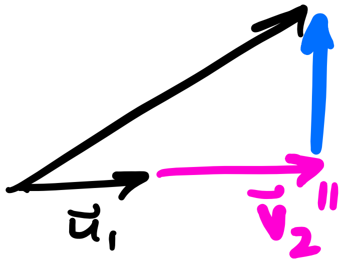
Idea:



$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$



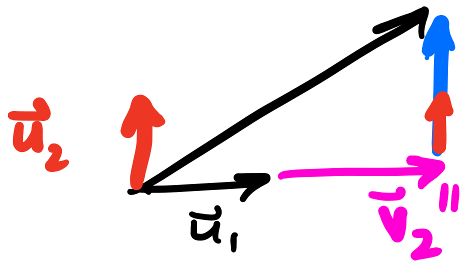
$\vec{v}_2^{\,\parallel} = \text{proj}_{\vec{u}_1} \vec{v}_2 = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$



$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

$$\vec{v}_2^{\parallel} = \text{proj}_{\vec{u}_1}\vec{v}_2 = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2\|^{\perp}}$$



$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

$$\vec{v}_2^{\parallel} = \text{proj}_{\vec{u}_1}\vec{v}_2 = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

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what about 3 vectors?

$\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \text{proj}_{\vec{u}_1}\vec{v}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$\vec{u}_2 = \vec{v}_2^{\perp} / \|\vec{v}_2\|$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \text{proj}_{\vec{u}_1}\vec{v}_3 - \text{proj}_{\vec{u}_2}\vec{v}_3$$

$$= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

$$\vec{u}_3 = \vec{v}_3^{\perp} / \|\vec{v}_3^{\perp}\|$$

Example: perform Gram-Schmidt on

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \right\}.$$

$$v_2 \cdot u_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= 3$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_1 = \vec{v}_1 / \|\vec{v}_1\| = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - (3)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2^{\perp} / \|v_2^{\perp}\| = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$$

$$= \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} - (5)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - (6)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

$$\vec{u}_3 = v_3^{\perp} / \|\vec{v}_3^{\perp}\| = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\vec{v}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\|\vec{v}_1\| = 5$$

$$\vec{u}_1 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$

$$\vec{v}_2^{\parallel} = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \left(-\frac{3}{5} + \frac{28}{5}\right)\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$

$$= 5\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2^{\perp} / \|\vec{v}_2^{\perp}\| = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}.$$