Orthogonal Projections and Orthonormal bases.

Def: Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^n$.

a) $\vec{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$

b) $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$

c) $\vec{u}$ is a unit vector if $\|\vec{u}\| = 1$.

any $\vec{v}$ has a corresponding parallel unit vector $\vec{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$.

A set of vectors $\vec{u}_1, \ldots, \vec{u}_m$ in $\mathbb{R}^n$ is called orthonormal if they are all unit vectors and all orthogonal to each other.

- $\vec{u}_j \cdot \vec{u}_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$

Example: $\vec{u}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

are an orthonormal set.

$$\|\vec{u}_1\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\|\vec{u}_2\| = \sqrt{(-\sin\theta)^2 + \cos^2\theta} = 1$$

$$\vec{u}_1 \cdot \vec{u}_2 = -\cos\theta\sin\theta + \cos\theta\sin\theta = 0.$$

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orthonormal vectors are linearly independent:

if $c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_m\vec{u}_m = \vec{0}$,

$$(c_1\vec{u}_1 + \ldots + c_m\vec{u}_m) \cdot u_1 = \vec{0} \cdot \vec{u}_1$$

$$c_1(\vec{u}_1 \cdot \vec{u}_1) + c_2(\vec{u}_2 \cdot \vec{u}_1) + \ldots + c_m(\vec{u}_m \cdot \vec{u}_1) = 0$$

$$c_1(1) + c_2(0) + \ldots + c_m(0) = 0$$

so $c_1 = 0$.

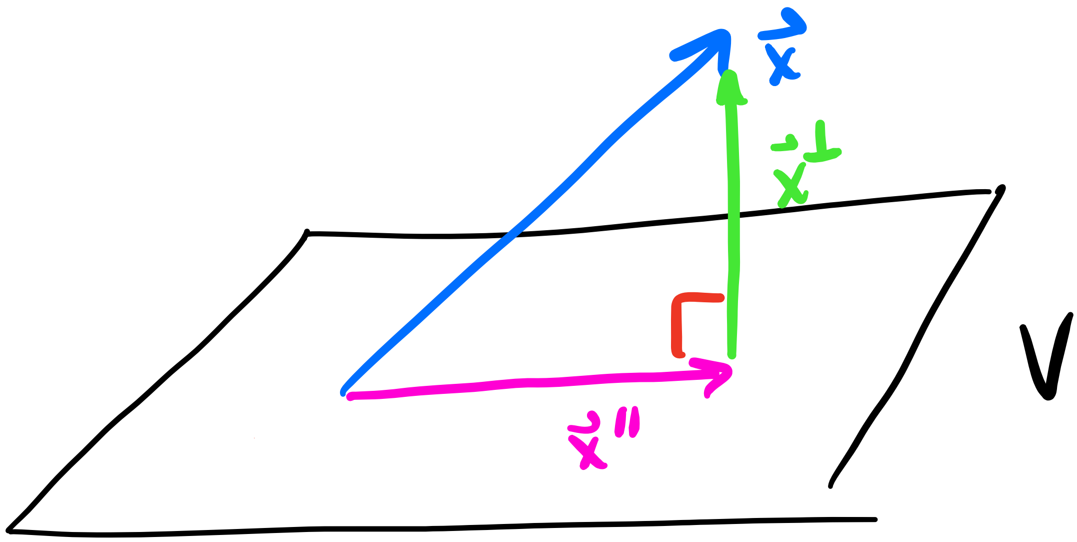
Same process shows <u>all</u> $c_i = 0$.

so the only relation is trivial.
so $\vec{u}_1, \dots, \vec{u}_n$ are linearly independent.

as a consequence, orthonormal vectors $\vec{u}_1, \dots, \vec{u}_n$ form a <u>basis</u> of $\mathbb{R}^n$

called an <u>orthonormal</u> basis.

Projection onto a subspace.



How can we find $\vec{x}^{\parallel} = \operatorname{proj}_V \vec{x}$?

If $\vec{u}_1, \dots, \vec{u}_m$ is an orthonormal basis for V,

$$\boxed{\operatorname{proj}_V \vec{x} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + \dots + (\vec{x}\cdot\vec{u}_m)\vec{u}_m} \quad !!!$$

Example: Let $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Find $\text{proj}_V \vec{x}$ for $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}$.

1. orthogonal basis?

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 0 \quad \checkmark$$

2. unit vectors? no.

$$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{u}_1 \cdot \vec{u}_2 = 0.$$

$$\vec{u}_1 \cdot \vec{u}_1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

$$\vec{u}_2 \cdot \vec{u}_2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1 \quad \checkmark$$

$$\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2$$

$$= \left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \\ 7 \end{pmatrix} \right) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \left(\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \\ 7 \end{pmatrix} \right) \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

Can check this by making sure

$\vec{x} - \text{proj}_V \vec{x} \perp \vec{u}_1$ and $\vec{u}_2$.

Consequence: If $\vec{u}_1, \dots, \vec{u}_n$ is an othronormal basis for $\mathbb{R}^n$,

$$\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_n \cdot \vec{x})\vec{u}_n$$

that is, if $\mathcal{U} = \{\vec{u}_1, \dots, \vec{u}_n\}$

$$[\vec{x}]_{\mathcal{U}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vdots \\ \vec{u}_n \cdot \vec{x} \end{bmatrix}.$$

Lingering Question:

How can we find (or make) an orthonormal basis?