Begin w/ invertible matrix thm.

include basis, span, lin ind.

3.4 Coordinates.

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

these are linearly independent,

so $\{\vec{v}_1, \vec{v}_2\}$ is a basis for

span $\{\vec{v}_1, \vec{v}_2\}$.

Lets give the basis a name:

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}.$$

Because $\mathcal{B}$ has two vectors,

dim $\{$ span $\{\vec{v}_1, \vec{v}_2\}\} = 2$.

and so $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ is a plane inside of $\mathbb{R}^3$.

is $\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ in the plane?

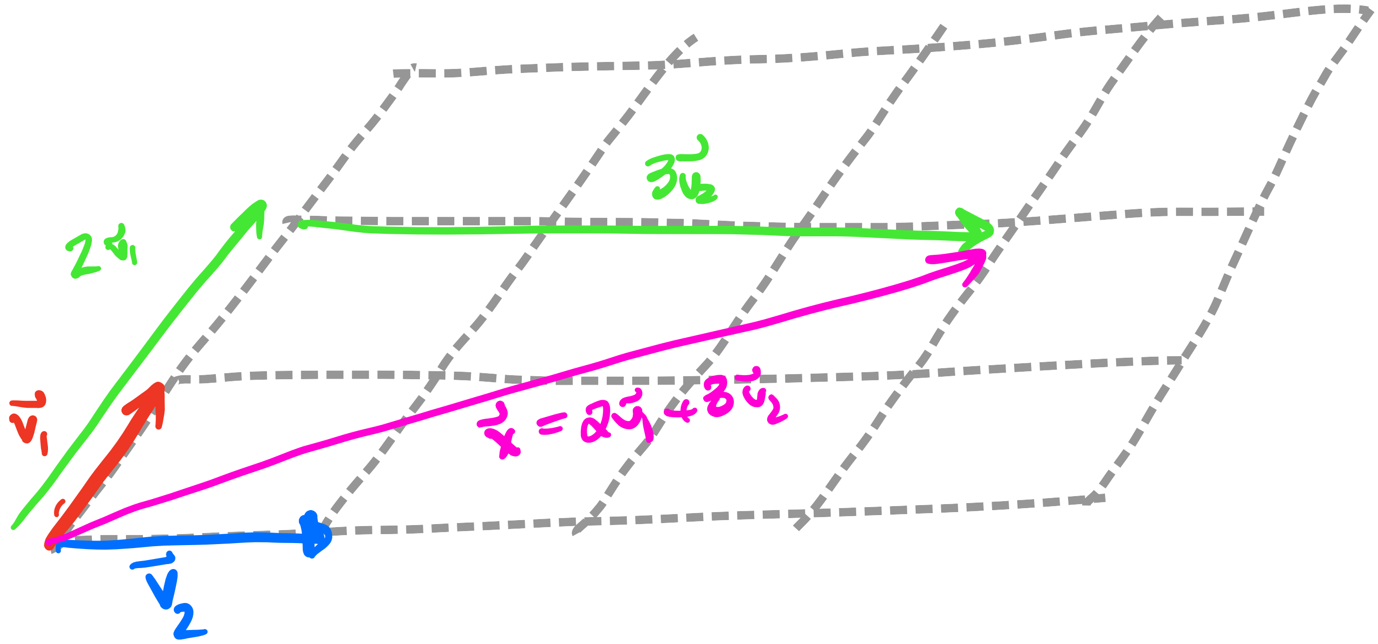
$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{x}$ <u>is</u> in $\text{span}\{\vec{v}_1, \vec{v}_2\}$.

$$\vec{x} = 2\vec{v}_1 + 3\vec{v}_2.$$

To visualize this, we can use a coordinate grid with axes parallel to $\vec{v}_1$ and $\vec{v}_2$.



the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ tells us the exact way to build the vector $\vec{x}$:

$$2\vec{v}_1 + 3\vec{v}_2.$$

Since we need to know the basis $\mathcal{B}$ for this to make sense, we write

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{B}} = 2\vec{v}_1 + 3\vec{v}_2.$$

$[\vec{x}]_B$ is called the <u>coordinate</u> <u>vector</u> of

$\vec{x}$ with respect to basis $\mathcal{B}$.

<u>Def</u> : $\boxed{\text{coordinates in a subspace.}}$

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a bases for a subspace V of $\mathbb{R}^n$.

Every vector $\vec{x}$ in V can be written <u>uniquely</u> as

$$\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_m.$$

$c_1, \dots, c_m$ are called the <u>coordinates</u> of $\vec{x}$ and

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{\mathcal{B}}$$ is the $\mathcal{B}$-coordinate vector of $\vec{x}$.

Hence $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$ means

$\vec{x} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$

Coordinates act exactly like subspaces

<u>Thm</u>: Suppose $\mathcal{B}$ is a basis for V.

1. $[\vec{x}+\vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$ for all x, y in V
2. $[k\vec{x}]_{\mathcal{B}} = k[x]_{\mathcal{B}}$ for all x in V, all k.

<u>Example</u>: Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$.

a. if $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, find $[\vec{x}]_{\mathcal{B}}$.

$$c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{\mathcal{B}} = \frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{\mathcal{B}}.$$

proof of linearity of coordinates:

suppose $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$.

$$\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

$$y = d_1\vec{v}_1 + \ldots + d_n\vec{v}_n$$

$$\vec{x}+\vec{y} = (c_1+d_1)\vec{v}_1 + \ldots + (c_n+d_n)\vec{v}_n$$

so $$[\vec{x}+\vec{y}]_B = \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$= [\vec{x}]_B + [\vec{y}]_B$$

Similar for $k[\vec{x}]_B = [k\vec{x}]_B$

b. If $[\vec{x}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{x}$.

$$\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} [\vec{x}]_B = \vec{x}$$

so $$\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}_B = \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \vec{x}$$

$$\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix} [\vec{x}]_B = \vec{x}$$

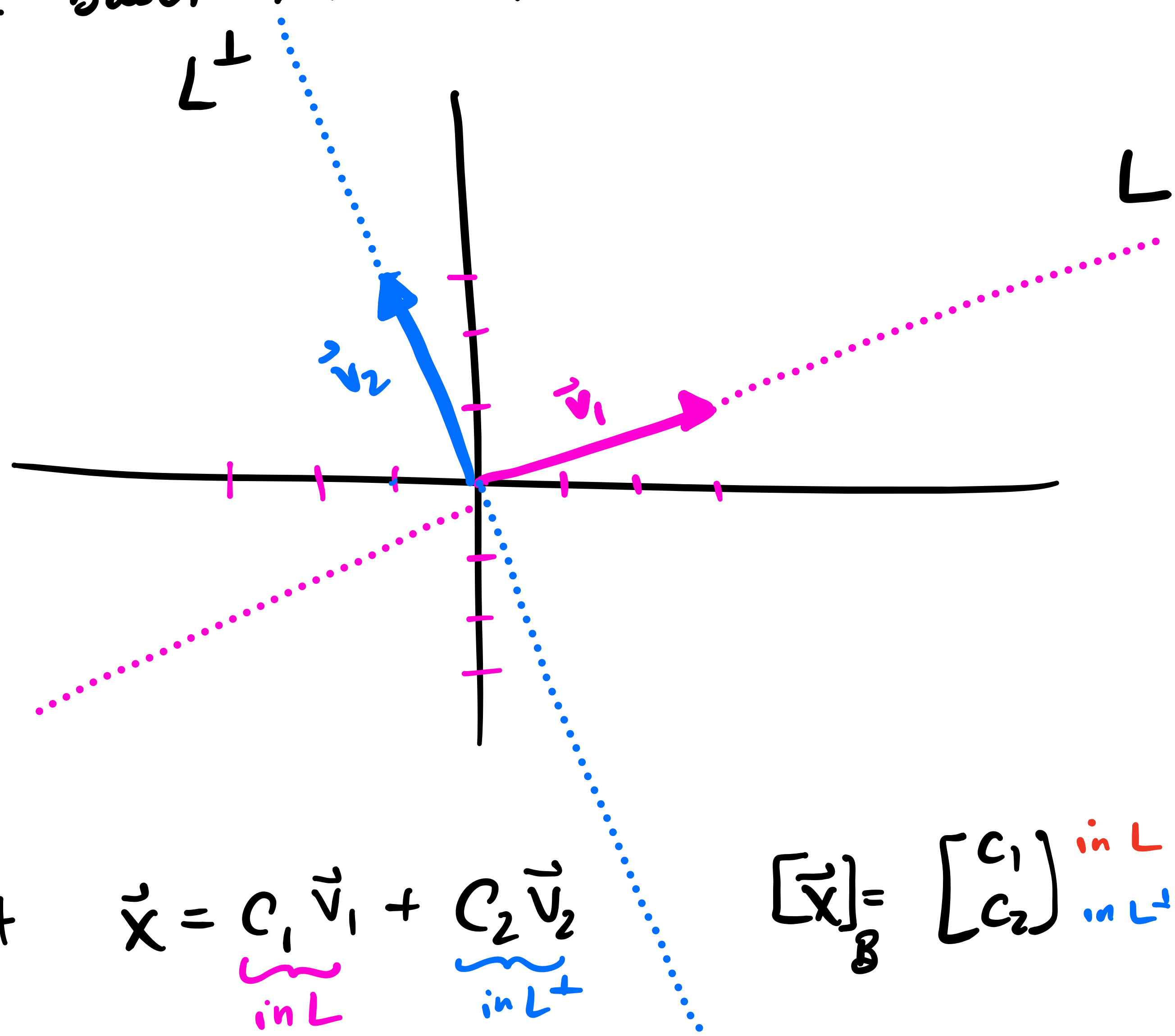
$$[\vec{x}]_B = [v_1 \cdots v_k]^{-1} \vec{x}$$

An application of coordinates

Let's look at how a change of basis affects a linear transformation.

The function T will not change, but the matrix representing T should.

Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ be a basis for $\mathbb{R}^2$.



Let $\vec{x} = \underbrace{c_1 \vec{v}_1}_{\text{in } L} + \underbrace{c_2 \vec{v}_2}_{\text{in } L^\perp}$ $\quad [\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \begin{matrix} \text{in } L \\ \text{in } L^\perp \end{matrix}$

Then $T(\vec{x}) = c_1 \vec{v}_1 = \text{proj}_L(\vec{x})$.

alternatively,

if $[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ then $[T(\vec{x})]_B = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$.

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the matrix that transforms $[\vec{x}]_B$ into $[T(\vec{x})]_B$.

$$[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [\vec{x}]_{\mathcal{B}}.$$

this is a very simple matrix.

So working in a new basis makes projection simple!

$$\begin{tikzcd}
\vec{x} = \overset{\text{in } L}{c_1\vec{v}_1} + \overset{\text{in } L^\perp}{c_2\vec{v}_2} \arrow[r, "T"] \arrow[d] & T(\vec{x}) = c_1\vec{v}_1 \arrow[d] \\
{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} \arrow[r, "B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}"'] & {[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}}
\end{tikzcd}$$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is called the $\mathcal{B}$-matrix of T or the matrix of T with respect to the basis $\mathcal{B}$.

Theorem: Consider a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and a basis $\mathfrak{B}$ of $\mathbb{R}^n$. There is a unique $n \times n$ matrix B so that

$$B[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{B}},$$

called the $\mathfrak{B}$-matrix of T.

$$B = \left[[T(v_1)]_{\mathfrak{B}} \cdots [T(v_n)]_{\mathfrak{B}} \right].$$

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pf: write $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$.

Then $[T(\vec{x})]_{\mathfrak{B}}$

$$= [T(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)]_{\mathfrak{B}}$$

$$= [c_1T(v_1) + \dots c_nT(\vec{v}_n)]_{\mathfrak{B}}.$$

$$= c_1[T(\vec{v}_1)]_{\mathfrak{B}} + \dots + c_n[T(v_n)]_{\mathfrak{B}}$$

$$= \left[[T(v_1)]_{\mathfrak{B}} \cdots [T(\vec{v}_n)]_{\mathfrak{B}} \right] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= B[\vec{x}]_{\mathfrak{B}}.$$

we already showed that

$$\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} [\vec{x}]_{\mathcal{B}} = \vec{x}$$

and

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}^{-1} \vec{x}.$$

Our projection onto L had

$\mathcal{B}$-matrix $\underset{B}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$.

in the standard basis, it has a different matrix:

$$\vec{u} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \qquad \text{proj}_L \vec{x} = \underset{A}{\begin{bmatrix} 9/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix}} \vec{x}.$$

what is the relationship between

A and B?

$$T(\vec{x}) = A\vec{x} \quad ?$$

$$[T(\vec{x})]_B = B [\vec{x}]_{\mathcal{B}}.$$

Even more more coordinates:

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}.$$

$$\boxed{T}$$

two ways to apply T

$$\vec{x} \longrightarrow A\vec{x} = T(\vec{x}) \qquad \mathcal{S} \text{ coords}$$

convert to $\mathcal{B}$-coords   apply B convert back to $\mathcal{S}$ coord

or
$$\underbrace{\vec{x} \xrightarrow{S^{-1}} [\vec{x}]_{\mathcal{B}}}_{\text{how do we do these maps.}} \xrightarrow{B} B[\vec{x}]_{\mathcal{B}} \underbrace{\xrightarrow{S} T(\vec{x})}_{\text{how?}}$$

$$B[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}$$

let $S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

$$S : [\vec{x}]_{\mathcal{B}} \to \vec{x}$$

$$S^{-1} : \vec{x} \longrightarrow [\vec{x}]_{\mathcal{B}}.$$

Since $A\vec{x} = T(\vec{x}) = SBS^{-1}x$,

$$A = SBS^{-1}$$

Shall we check?

$$\begin{bmatrix} 9/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix},$$

$$\underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{B} \underbrace{\left(\frac{1}{10}\right) \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}}_{S^{-1}}$$

$$= \frac{1}{10} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 9/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix} !!!$$

Two square matrices are called <u>similar</u> if $A = S^{-1}BS$.

that is $T(\vec{x}) = A\vec{x}$ and $R(\vec{x}) = B\vec{x}$
differ by a change of bases.

Find a basis for which

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$ is similar to a diagonal

matrix.

$r_4, 1$

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

note that

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$= 4\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2}c_2 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and $$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$\vec{x} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\left[T\begin{bmatrix}3\\2\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}4\\0\end{bmatrix}$$

$$T\begin{bmatrix}1\\-1\end{bmatrix} = (-1)\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\left[T\begin{bmatrix}1\\-1\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}0\\-1\end{bmatrix}.$$

so $B = \begin{bmatrix}4&0\\0&-1\end{bmatrix}.$

and $$\underbrace{\begin{bmatrix}2&3\\2&1\end{bmatrix}}_{T} = \underbrace{\underset{\mathcal{S}\leftarrow\mathcal{B}}{\begin{bmatrix}3&1\\2&-1\end{bmatrix}}\ \underset{[T]_{\mathcal{B}}}{\begin{bmatrix}4&0\\0&-1\end{bmatrix}}\ \underset{\mathcal{B}\leftarrow\mathcal{S}}{\left(-\frac{1}{5}\begin{bmatrix}-1&-1\\-2&3\end{bmatrix}\right)}}_{T}$$

$$\begin{tikzcd}
\vec{x} \arrow[r, "\begin{bmatrix}2&3\\2&1\end{bmatrix}"] \arrow[d, "S^{-1}"'] & T(\vec{x}) \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "\begin{bmatrix}4&0\\0&-1\end{bmatrix}"'] & {[T(x)]_{\mathcal{B}}} \arrow[u, "S"']
\end{tikzcd}$$