# Subspaces and dimension

**Thm:** All bases of a subspace $V$ of $\mathbb{R}^n$ contain the same number of vectors.

**Def:** The <u>dimension</u> of a subspace $V$ is the number of vectors in a basis for $V$.

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Let's look at some consequences.

Consider $V \subseteq \mathbb{R}^n$ with $\dim(V) = m$

a. We can find <u>at most</u> $m$ linearly independent vectors in $V$.

b. <u>At least</u> $m$ vectors are needed to span $V$.

c. If $m$ vectors in $V$ are linearly independent, they span $\bar{V}$.

d. If $n$ vectors span $\bar{V}$, they are a basis for $V$.

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Example: Basis of kernel and image

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$

Find a basis for $\ker A$.

Solve $A\underset{\sim}{x} = 0$

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

$x_2, x_4, x_5$ free

$$\begin{aligned} x_2 &= \alpha & \qquad x_1 &= -2x_2 - 3x_4 + 4x_5 \\ x_4 &= \beta & &= -2\alpha - 3\beta + 4\gamma. \\ x_5 &= \gamma & x_3 &= 4x_4 - 5x_5 \\ & & &= 4\beta - 5\gamma. \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2\alpha - 3\beta + 4\gamma \\ \alpha \\ 4\beta - 5\gamma \\ \beta \\ \gamma \end{bmatrix}$$

$$= \alpha \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{w}_1} + \beta \underbrace{\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}_2} + \gamma \underbrace{\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_3}.$$

$\vec{w}_1, \vec{w}_2, \vec{w}_3$ span $\ker A$ as every solution to $A\vec{x}=0$ is $\vec{x} = \alpha\vec{w}_1 + \beta\vec{w}_2 + \gamma\vec{w}_3$.

also, $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are linearly independent by the $\bigcirc$ argument

hence $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a basis for $\ker A$.

$\dim \ker(A) = 3$

Find a basis for $\operatorname{im}(A)$.

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & 7 \end{bmatrix}$$

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad \vec{a}_4 \quad \vec{a}_5$

$$\text{rref}(A) = B = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{a}_1$ is not redundant $\qquad$ $\vec{a}_3$ is not redundant

So $\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\} = \text{span}\{\vec{a}_1, \vec{a}_3\}$.

and $\{\vec{a}_1, \vec{a}_3\}$ is a linearly independent set.

So $\text{im}(A)$ has $\{\vec{a}_1, \vec{a}_3\}$ as a basis

$$\dim \text{im}(A) = 2$$

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The argument we just used shows

$$\dim(\text{im}(A)) = \text{rank } A.$$

$\dim(\ker A) = \#$ of free variables

$= \#$ of columns $- \#$ of leading variables

$= m - \text{rank}(A).$

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so $\underbrace{\dim(\ker(A))}_{\text{nullity}(A)} + \underbrace{\dim(\text{im}(A))}_{\text{rank}(A)} = m$

This idea is so important, it is sometimes called the fundamental theorem of linear algebra.

**Thm**: $\boxed{\text{rank-nullity theorem}}$

For any $n \times m$ matrix $A$,

$$\dim(\ker(A)) + \dim(\text{im}\, A) = m.$$

or

$$\text{nullity}(A) + \text{rank}(A) = m.$$

Sometimes we can use this to construct bases for ker A and im A by inspection.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}$$

by observation, $\vec{v}_2 = 2\vec{v}_1$, $\vec{v}_3 = \vec{0}$, $\vec{v}_5 = \vec{v}_1 + \vec{v}_4$.

$\vec{v}_1, \vec{v}_4$ not redundant,

so $\{\vec{v}_1, \vec{v}_4\}$ is a basis for $\mathrm{im}(A)$

$\dim(\mathrm{im}(A)) = 2.$

so $\dim(\ker(A)) = 5 - 2 = 3.$

Now find three vectors in $\ker(A)$

use relations.

$-2\vec{v}_1 + \vec{v}_2 = 0$ $\qquad \vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_3 = 0$ $\qquad \vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$-\vec{v}_1 - \vec{v}_4 + \vec{v}_5 = \vec{0}$ $\qquad \vec{w}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$