<u>Def</u>: A subset W of $\mathbb{R}^n$ is called a $\boxed{\text{subspace}}$ if

a. W contains the $\vec{0}$ vector.

b. W is closed under addition
$\vec{w}_1$ and $\vec{w}_2$ in W $\Rightarrow$ $\vec{w}_1 + \vec{w}_2$ in W.

c. W is closed under scalar multiplication
$\vec{w}$ in W $\Rightarrow$ $k\vec{w}$ in W for all scalars k.

basically, a subspace is a set that is closed under linear combination.

<u>Thm</u>: If $T(\vec{x}) = A\vec{x}$, $T: \mathbb{R}^m \to \mathbb{R}^n$
then
- $\ker(T)$ is a subspace of $\mathbb{R}^m$
- $\mathrm{im}(T)$ is a subspace of $\mathbb{R}^n$.

Example: is $W = \left\{ \begin{bmatrix} 1 \\ y \end{bmatrix} : y \text{ in } \mathbb{R} \right\}$

a subspace?

<u>No</u> $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in W.

Example: is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 \quad y \geq 0 \right\}$.

a subspace?

<u>No</u>. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in W.

$(-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ <u>not</u> in W.

not closed under scalar mult.

Example: is $W = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \text{ in } \mathbb{R} \right\}$

a subspace?

a. $\vec{0}$ in W. ✓

b. $\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{pmatrix} x+y \\ 0 \end{pmatrix}$ in W.

e. $k\begin{bmatrix}x\\0\end{bmatrix} = \begin{bmatrix}kx\\0\end{bmatrix}$ in W.

yes

Every vector in $\mathbb{R}^2$ has the form

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\0\end{bmatrix} + \begin{bmatrix}0\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$$

$$= x\vec{i} + y\vec{j}.$$

every vector in $\mathbb{R}^2$ then is a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ are sort of a "skeleton" of $\mathbb{R}^2$, as $\mathbb{R}^2 = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$.

We want to extend this idea to other subspaces.

Consider $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

$$\operatorname{im}(A) = \operatorname{span}\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}} \right\}.$$

can we use fewer vectors?

Since $\vec{v}_2 = 2\vec{v}_1$, it contributes nothing new to a linear combination.

also,

$$\vec{v}_4 = \vec{v}_1 + \vec{v}_3.$$

so $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$

$= c_1\vec{v}_1 + 2c_2\vec{v}_1 + c_3\vec{v}_3 + c_4(\vec{v}_1 + \vec{v}_3)$

$= (c_1 + 2c_2 + c_4)\vec{v}_1 + (c_3 + c_4)\vec{v}_3$

any lin combination of $\vec{v}_1, \dots \vec{v}_4$ is really just a lin combination of $\vec{v}_1$ and $\vec{v}_3$.

$\vec{v}_2$ and $\vec{v}_4$ are called

redundant vectors,

as they are linear combination of previous vectors.

So $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \text{span}\{\vec{v}_1, \vec{v}_3\}$.

extra information — redundant vectors

exactly enough information — no redundant vectors.

Definition. Let $\vec{v}_1, \dots, \vec{v}_n$ be a list of vectors in $\mathbb{R}^m$.

- A vector $\vec{v}_i$ is called redundant if $\vec{v}_i$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}$.

- $\vec{v}_1, \dots, \vec{v}_n$ are called <u>linearly independent</u> if none of them is redundant. Otherwise, the set is called <u>linearly dependent</u>.

- We say $\vec{v}_1, \dots, \vec{v}_n$ in a subspace W is a <u>basis</u> for W if
  1. $W = \text{span}\{\vec{v}_1 \dots \vec{v}_n\}$.
  2. $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

<u>In</u> the previous example,

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix},$$

we discovered $\text{im}(A) = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$.

also $\begin{bmatrix}1\\2\\3\end{bmatrix} \neq k_1 \begin{bmatrix}1\\1\\1\end{bmatrix}$ for any k.

so $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$ is linearly independent.

Thm: To form a basis for im(A),

- list all columns of A
- remove redundant vectors.

But how?

Example: are $\overset{\vec{v}_1}{\begin{bmatrix}1\\2\\3\end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix}4\\5\\6\end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix}7\\8\\9\end{bmatrix}}$ linearly independent?

$\vec{v}_1, \vec{v}_2$ are not redundant, since $\vec{v}_2 \neq c\vec{v}_1$ for any c.

is $\vec{v}_3$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

is $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}_3$?

$$\begin{pmatrix}\vec{v}_1 & \vec{v}_2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \vec{v}_3 \ ?$$

<u>row reduce</u>!

$$\left(\begin{array}{cc|c}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{array}\right) \sim \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\end{pmatrix}$$

$c_1 = -1 \quad c_2 = 2.$

So yes! $(-1)\vec{v}_1 + (2)\vec{v}_2 = \vec{v}_3$

if we rewrote this as a homogeneous equation,

$$\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$$

This kind of an equation is called a linear relation between $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

DEF: Let $\vec{v}_1, \dots, \vec{v}_n$ be vectors in $\mathbb{R}^n$.

An equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$$ is

called a linear relation between $\vec{v}_1 \dots \vec{v}_n$.

The trivial relation is $c_1 = c_2 = \dots = c_n = 0$.

A nontrivial relation has at least one $c_i \neq 0$.

This may or may not exist.

Thm: $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent if and only if there exist nontrivial relation between them.

How can we find non trivial relations?

Ex: Recall $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

$$(1)\vec{v}_1 + (-2)\vec{v}_2 + (1)\vec{v}_3 = \vec{0}.$$

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \vec{0}.$$

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is in $\ker(A)$, $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

Is there an easier way to do this?

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

what does this tell us?

$\vec{v}_2 = 2\vec{v}_1$. $\qquad$ $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$.

$2\vec{v}_1 - \vec{v}_2 = 0$ $\qquad$ $\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = 0$

these are called <u>linear relations</u>