3.1 Image and Kernel

For a linear transformation $T : X \to Y$

the set X is the domain of T,

the set Y is the codomain of T.

Definition The image of T is the set $\text{image}(T) = \{T(\vec{x}) : \vec{x} \in X\}$.

$$= \{\vec{b} \in Y : \vec{b} = T(\vec{x}) \text{ for some } \vec{x} \in X.\}$$

we might also call this the range of T

For example, suppose $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$. for $T: \mathbb{R}^2 \to \mathbb{R}^2$.

$\text{image}\, T = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$.

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$= (x_1 + 3x_2)\begin{bmatrix}1\\2\end{bmatrix}.$

So every output of T is parallel to $\begin{bmatrix}1\\2\end{bmatrix}$.

So $\text{image}(T) = \left\{ c\begin{bmatrix}1\\2\end{bmatrix} : c \in \mathbb{R} \right\}$.

Crucial Definition:

Given vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$, the set of all linear combinations of $\vec{v}_1 \dots \vec{v}_n$ is called the span and is denoted

$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = \{ c_1\vec{v}_1 + \dots + c_n\vec{v}_n : c_1, \dots, c_n \in \mathbb{R} \}.$$

Theorem: The image of $T(\vec{x}) = A\vec{x}$ is the span of the column vectors of A.

Pf: Suppose $T(\vec{x}) = \begin{pmatrix} | & | & & \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$

$$= x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m$$

So every output of T is a linear combination of the columns of A and nice version

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Theorem: For $T : \mathbb{R}^n \to \mathbb{R}^n$ linear, $\mathrm{im}(T)$ has these important properties.

① $T(\vec{0}) = \vec{0}$, so $\vec{0}$ is in $\mathrm{im}(T)$.

② if $\vec{b}, \vec{c}$ in $\mathrm{im}(T)$,

$\vec{b} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ for $c_1, \dots, c_m$ in $\mathbb{R}$

$\vec{c} = d_1\vec{v}_1 + \dots + d_m\vec{v}_m$ for $d_1 \dots d_m$ in $\mathbb{R}$

then $\vec{b} + \vec{c} = (c_1\vec{v}_1 + \dots + c_m\vec{v}_m) + (d_1\vec{v}_1 + \dots + d_m\vec{v}_m)$

$= (c_1 + d_1)\vec{v}_1 + \dots + (c_m + d_m)\vec{v}_m$

in $\mathrm{im}(T)$.

$\mathrm{im}(T)$ is closed under vector addition

③ If $\vec{b}$ in $\mathrm{im}(T)$,

$\vec{b} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$.

then $k\vec{b} = k(c_1\vec{v}_1 + \dots + c_m\vec{v}_m)$

$= (kc_1)\vec{v}_1 + \dots + (kc_m)\vec{v}_m$.

in $\mathrm{im}(T)$

im(T) is closed under scalar multiplication.

These will, later defined to be the properties that characterize subspaces.

The set of vectors that T sends to 0 is special.

<u>Definition</u>: the <u>kernel</u> (or null space) of $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ is the set

$$\{x \text{ in } \mathbb{R}^m : T(\vec{x}) = \vec{0}\}$$

$$= \{\vec{x} : A\vec{x} = 0\}.$$

im(T) lives in the codomain or target space

ker(T) lives in the domain or initial space.

Example: Let $T\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\vec{x}$.

Find im(T) and ker(T).

$$\text{im}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}.$$

$$\ker(T) = \text{solutions to } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$x_1 = \alpha$

$x_2 = -2\alpha.$

$x_3 = \alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ -2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

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properties of kernel:

① $\vec{0}$ in $\ker(T)$ $\quad (T(0) = 0)$

② if $\vec{u}, \vec{v}$ in $\ker(T)$, $\vec{u} + \vec{v}$ in $\ker(T)$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{0} + \vec{0} = \vec{0}.$$

③ if $\vec{u} \in \ker(T)$,

$$T(k\vec{u}) = kT(\vec{u}) = k\vec{0} = \vec{0}$$

so $k\vec{u}$ in $\ker T$.

How about invertible matrices?

if A is invertible,

$T(\vec{x}) = Ax$ has ker(T) solution to

$$A\vec{x} = 0.$$

but in this case, the only solution is $\vec{0}$.

So If A is invertible, $\ker(T) = \{\vec{0}\}$.

alternatively, <u>every</u> $\vec{b}$ in $\mathbb{R}^n$ has a solution to

$A\vec{x} = b$, so every $\vec{b}$ in $\mathrm{im}(T)$.

$\mathrm{im}(T) = \mathbb{R}^n \Leftrightarrow T$ is invertible.

<u>A is $n \times n$</u>

- A is invertible
- $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$.
- $A\vec{x} = \vec{0}$ is solved only by $\vec{x} = 0$
- $\mathrm{rref}(A) = I_n$.

- $\operatorname{rank}(A) = n$
- $\operatorname{im}(A) = \mathbb{R}^n$
- $\ker(A) = \{\vec{0}\}$.