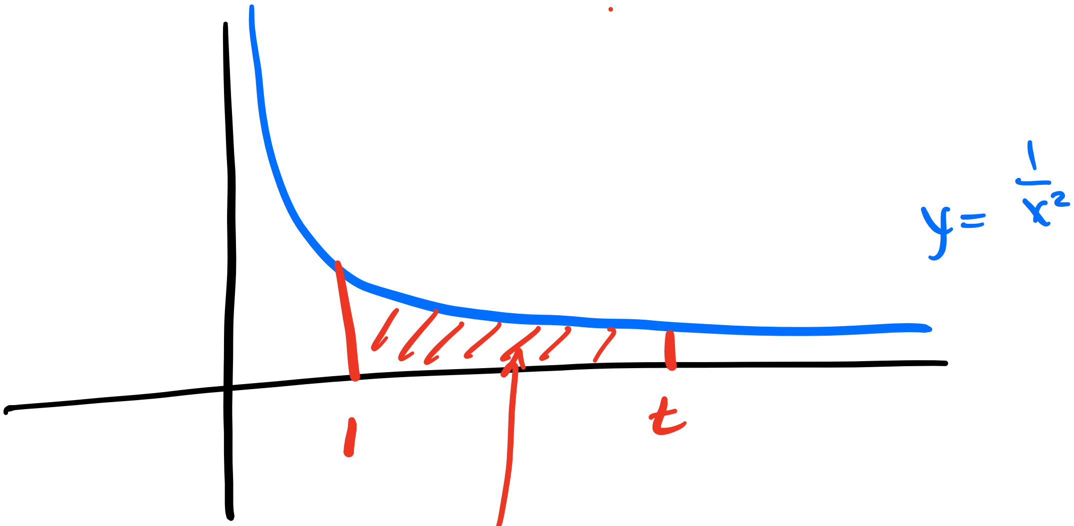
Improper integrals: 7.8

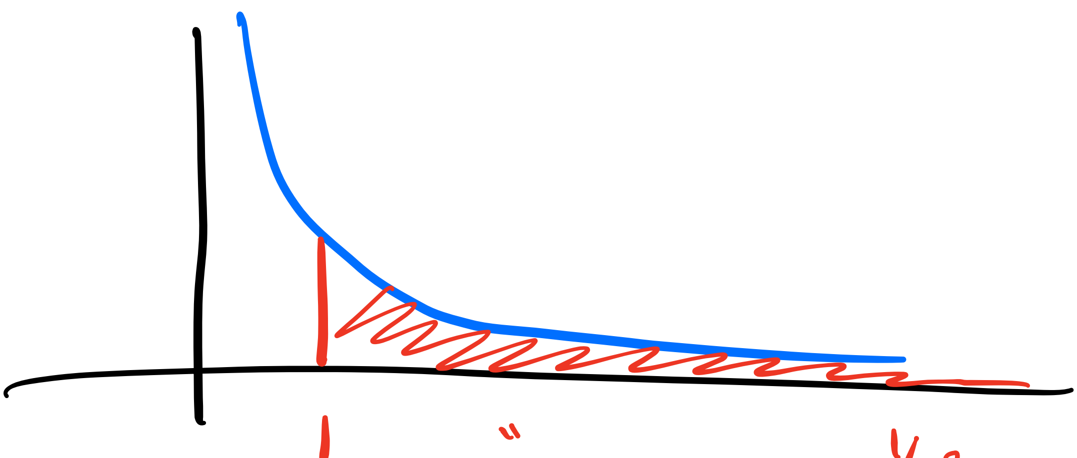


$$\int_1^t \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\Big|_1^t = 1-\frac{1}{t}$$

what happens as $t\to\infty$?

$$\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1-\frac{1}{t} = 1-0=1$$



"area under $1/x^2$ on $(1,\infty)$"?

So we ~~define~~ define the symbol

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ to mean } \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx.$$

The FTC **DOES NOT APPLY**

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- (a,b) must be finite (no ∞)
- f has to be piecewise continuous. (no vert asy.)

$$\int_1^\infty \frac{1}{x^2}\,dx \cancel{=} -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty} + 1$$

noooooo

This is a dangerous habit. Don't do it!

Another example:

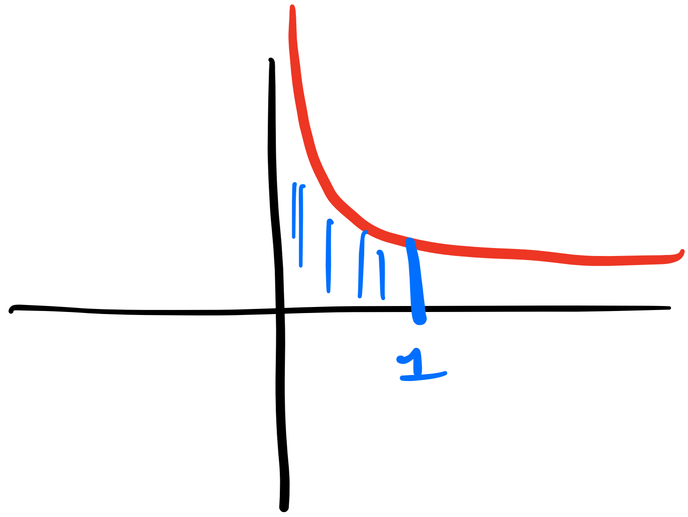
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to 0^+} 2\sqrt{x}\,\Big|_t^1$$

$$= \lim_{t\to 0^+} 2\cdot - 2\sqrt{t}$$

$$= 2$$



$f(x) = \frac{1}{\sqrt{x}}$ isn't defined at 0!

can't integrate at an asymptote.

$$\int_0^1 \frac{1}{x^2}\,dx$$

$$= \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} -\frac{1}{x}\,\Big|_t^1$$

$$= \lim_{t\to 0} \frac{1}{t} - 1 \quad \infty.$$

DNE

Type I improper integrals

DEF: $\int_a^\infty f(x)\,dx$

1. $\int_a^t f(x)\,dx$ exists for all $t > a$.
2. $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx$ exists.

then we say $\int_a^\infty f(x)\,dx = \lim\limits_{t\to\infty} \int_a^t f(x)\,dx$

converges.

If $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx$ DNE, it <u>diverges</u>.

Likewise,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx.$$

converges if the limit exists.

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{c} f(x)\,dx + \int_c^\infty f(x)\,dx}_{\text{both must converge independently}} \text{ for any } c.$$

for this to converge.

Example:

$$\int_2^\infty \frac{1}{x}\,dx$$

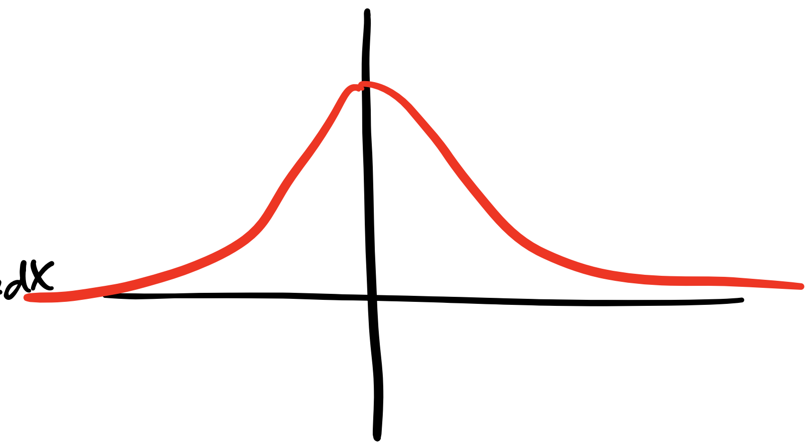
$$= \lim_{t\to\infty} \int_2^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \ln x \Big|_2^t$$

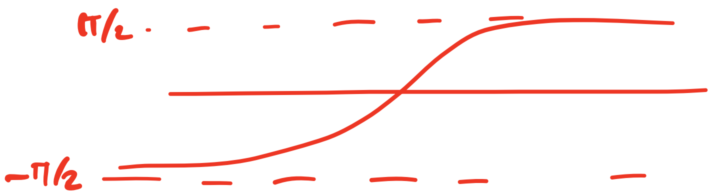
$$= \lim_{t\to\infty} \ln t - \ln 2 = \infty. \quad \underline{\text{divergent}}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$



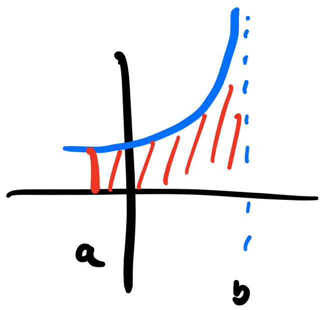
$$= \lim_{A\to-\infty} \arctan(0) - \arctan(A) + \lim_{B\to\infty} \arctan(B) - \arctan(0).$$



$$= 0 - \left(\frac{-\pi}{2}\right) + \frac{\pi}{2} + 0 = \pi.$$

Type I deals with horizontal asymptotes.
How about vertical?

Type II: from left:

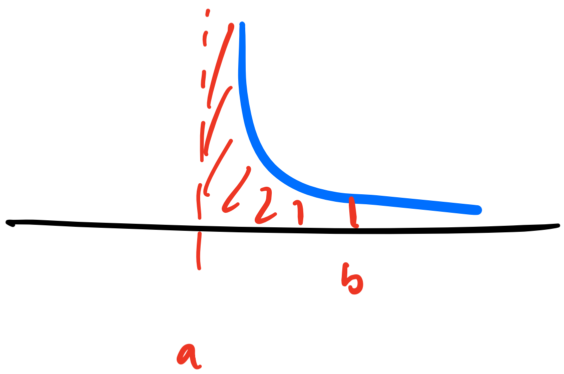


$$\int_a^b f(x)\,dx$$

$$= \lim_{t \to b^-} \int_a^t f(x)\,dx$$

If the limit exists, this is convergent.

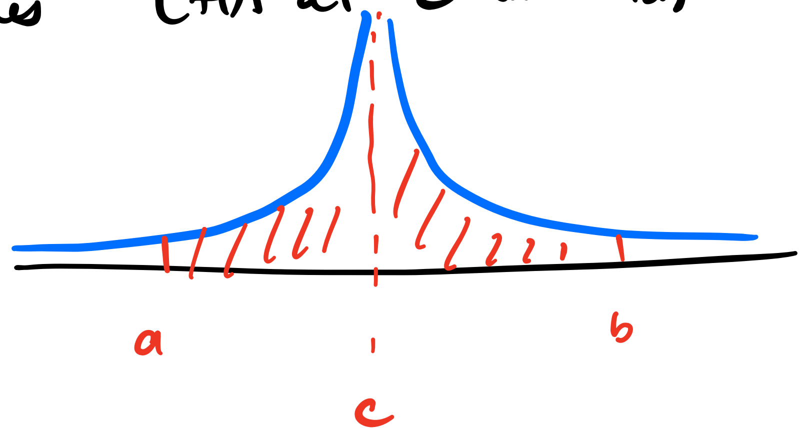
from right:



$$\int_a^b f(x)\,dx$$

$$= \lim_{t \to a^+} \int_t^b f(x).$$

both sides (HA at c in (a,b))



$$\int_a^b f(x)\,dx$$

$$= \lim_{R \to c^+} \int_a^R f(x)\,dx + \lim_{L \to c^-} \int_L^b f(x)\,dx.$$

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these can be harder to see.

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx \qquad \text{asymptote at } x=2$$

improper!

$$\lim_{t \to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t \to 2^+} 2\sqrt{x-2}\,\Big|_t^5 = \lim_{t \to 2^+} 2\sqrt{3} - \cancel{2\sqrt{t-2}}^{\,0}$$

$$= 2\sqrt{3}.$$

Be Careful

$$\int_0^3 \frac{1}{x-1}\,dx$$

NO

$$\ln|x-1| \Big|_0^3$$

$$= \ln|3-1| - \ln|0-1|$$

$$= \ln|2| - \ln|1|$$

$$= \ln 2.$$