partial fractions 7.4

A rational function has the form

$f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Every polynomial factors into

linear terms : $(ax+b)$

irreducible quadratic terms : (ax^2+bx+c)

b^2-4ac is the discriminant and tells us if a quadratic can be factored.

Example: $x^3-1 = \underbrace{(x-1)}_{\text{linear}}\underbrace{(x^2+x+1)}_{\text{irreducible quadratic}}$

A rational function is called <u>proper</u> if the highest power in the numerator (or degree) is less than the degree of the denominator.

<u>Ex</u>: $\frac{x}{x^2+1}$ is proper. (deg 1 / deg 2)

$\frac{x^2-1}{x^2+x+1}$ is <u>not</u> proper (deg 2 / deg 2)

If a rational function is improper, we can use polynomial long division to reduce it to a proper rational function.

$$\frac{x^2-1}{x^2+x+1}$$

$$\begin{array}{r l}
 & 1 + \frac{-x-2}{x^2+x+1} \\
x^2+x+1 \overline{)} & x^2+0x-1 \\
\text{sub} & x^2+x+1 \\
\hline
 & 0 \ \boxed{-x-2} \text{ remainder}
\end{array}$$

so $\frac{x^2-1}{x^2+x+1} = 1 + \frac{-x-2}{x^2+x+1}$.

Ex: $\frac{x^3-1}{x+2}$

$=$

$x^2-2x+4-\frac{9}{x+2}$

$$\begin{array}{r l}
 & x^2-2x+4 - \frac{9}{x+2} \\
x+2 \overline{)} & x^3+0x^2+0x-1 \\
\text{sub.} & x^3+2x^2 \\
\hline
 & 0 \ -2x^2+0x-1 \\
\text{sub.} & -2x^2-4x \\
\hline
 & 0 \ +4x-1 \\
\text{sub} & 4x+8 \\
\hline
 & 0 \ -9 \text{ remainder}
\end{array}$$

we'll show by example how a proper rational function breaks apart into <u>partial fractions</u> (like $\frac{A}{x+3}$ or $\frac{Bx+C}{x^2+1}$)

Case I – simple linear terms:

$$f(x) = \frac{1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

linear terms break up into fractions like these

Case II – repeated linear terms

$$f(x) = \frac{1}{(x+2)^3(x-1)} = \underbrace{\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}} + \frac{D}{x-1}$$

one partial fraction for each power of the repeated term.

Case III – single irreducible quadratic

$$f(x) = \frac{x}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

irreducible quadratics get linear numerators.

Case IV - repeated quadratics

$$f(x) = \frac{1}{x(x^2+4)^2} = \frac{A}{x} + \underbrace{\frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}}$$

one partial fraction for each power.

Finding the constants A, B, ...

① partial fraction decomposition

$$\frac{x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

② clear denominators

$$(x-3)(x+1)\frac{x}{(x-3)(x+1)} = \left(\frac{A}{x-3} + \frac{B}{x+1}\right)(x-3)(x+1)$$

$$x = A(x+1) + B(x-3)$$

③ plug in useful values.

$x = -1$

$$-1 = A(0) + B(-4)$$

$$B = 1/4$$

$x = 3$

$$3 = A(4) + B(0)$$

$$A = \frac{3}{4}$$

(4) compare coefficients if any information remains to be found. N/A.

Example integration

$$\int \frac{x}{(x-3)(x+1)}\,dx \overset{PFD}{=} \int \frac{3/4}{x-3} + \frac{1/4}{x+1}\,dx$$

$$= \frac{3}{4}\int \frac{1}{x-3}\,dx + \frac{1}{4}\int \frac{1}{x+1}\,dx$$

$$= \frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + C$$

Example: $\int \frac{1}{x(x^2+1)}\,dx$

PFD: $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$$x(x^2+1)\frac{1}{x(x^2+1)} = \left(\frac{A}{x} + \frac{Bx+C}{x^2+1}\right)x(x^2+1)$$

Clear denominators

$$1 = A(x^2+1) + (Bx+C)x$$

$x=0$ plugin.

$$1 = A(1) + 0$$

$$A = 1$$

$$1 = 1(x^2+1) + (Bx+C)x.$$ compare coefficients

$$-x^2 = Bx^2 + Cx$$

$$-1x^2 + 0x = Bx^2 + Cx$$

$B = -1$
$C = 0$

So $$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}.$$

$$\int \frac{1}{x(x^2+1)}dx = \int \frac{1}{x} - \frac{x}{x^2+1}dx$$

$$= \int \frac{1}{x}dx - \int \frac{x}{x^2+1}dx$$

basic form. u-sub

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + C.$$

Example:

Given that $$\frac{x+2}{(x+1)(x^2+1)} = \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x + \frac{3}{2}}{x^2+1}$$

Calculate

$$\int \frac{x+2}{(x+1)(x^2+1)}dx.$$

$$= \int \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x + \frac{3}{2}}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{3}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{3}{2}\arctan x + C$$