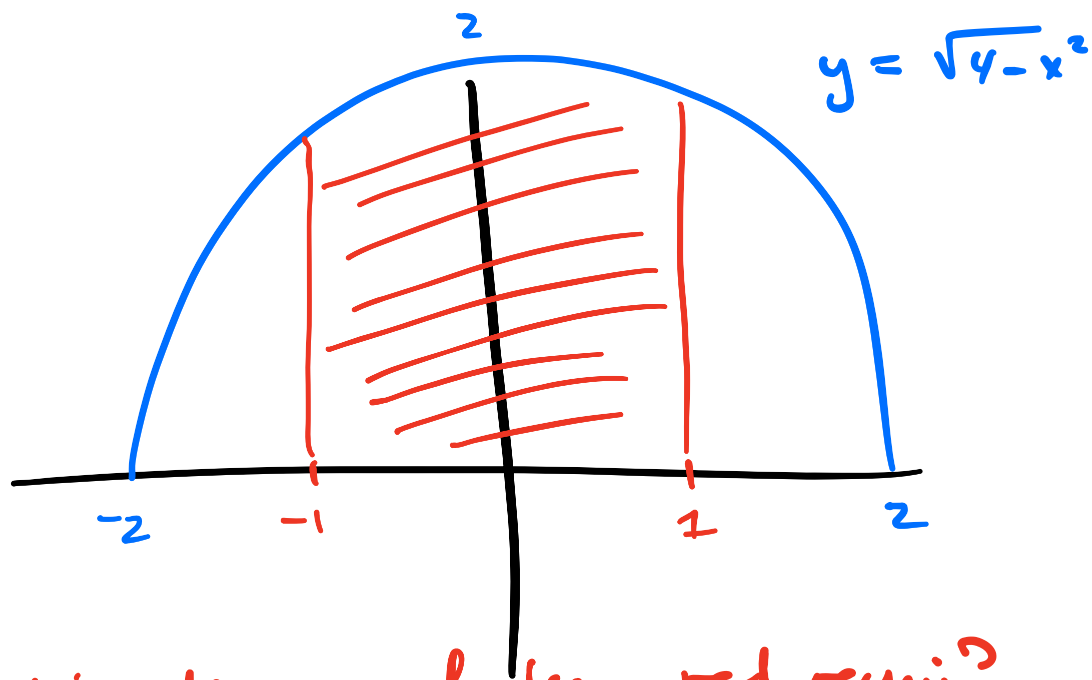
7.3 Trig substitution

Consider the following problem:



what is the area of the red region?

$$A = \int_{-1}^{1} \sqrt{4-x^2}\,dx$$

no u-sub available (unlike $\int x\sqrt{4-x^2}\,dx$)

so what to do? can we get rid of the square root?

$$\sqrt{4-x^2} \longrightarrow \sqrt{\square^2} = \square$$

kind of looks like a trig identity.

if $x = 2\sin u$,

then $\sqrt{4-(2\sin u)^2}$

$= \sqrt{4-4\sin^2 u}$

$= \sqrt{4(1-\sin^2 u)}$

$= \sqrt{4\cos^2 u}$

$= 2\cos u$!

lets give it a shot.

$$\int_{-1}^{1} \sqrt{4-x^2}\, dx$$

$x = 2\sin u$

$dx = 2\cos u\, du$

$u = \arcsin\left(\frac{x}{2}\right)$

$u(1) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$u(-1) = \frac{-\pi}{6}$

$$= \int_{-\frac{\pi}{6}}^{\pi/6} \sqrt{4-(2\sin u)^2}\; 2\cos u\, du$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\cos u\, 2\cos u\, du$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 4\cos^2 u\, du = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 4\left(\frac{1}{2}+\frac{1}{2}\cos 2u\right) du$$

$$= 2u + \sin 2u \Big|_{-\pi/6}^{\pi/6}$$

$$= \left(\frac{\pi}{3} + \sin\frac{\pi}{3}\right) - \left(-\frac{\pi}{3} + \sin\frac{-\pi}{3}\right)$$

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how to identify what to do:

$1-\sin^2 u = \cos^2 u$	$\sqrt{a^2-x^2}$	$x = a\sin u$
$\tan^2 u + 1 = \sec^2 u$	$\sqrt{x^2+a^2}$	$x = a\tan u$
$\sec^2 u - 1 = \tan^2 u$	$\sqrt{x^2-a^2}$	$x = a\sec u$

Example:

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$

$x = 3\sin u$

$dx = 3\cos u\,du$

$$= \int \frac{\sqrt{9-9\sin^2 u}}{9\sin^2 u}\,(3\cos u\,du)$$

$$= \int \frac{3\cos u\;3\cos u}{9\sin^2 u}\,du$$

$$= \int \cot^2 u\,du$$

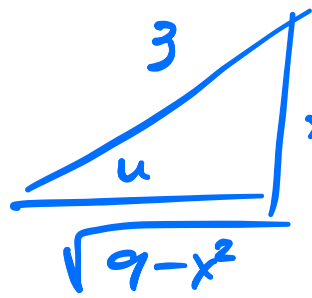
$$= \int \csc^2 u - 1 \, du \qquad x = 3\sin u, \quad u = \arcsin\left(\tfrac{x}{3}\right)$$

$$= -\cot u - u + C$$

$$= -\cot\left(\arcsin\left(\tfrac{x}{3}\right)\right) - \arcsin\left(\tfrac{x}{3}\right) + C$$

$\sin u = \frac{x}{3}$

$$= \frac{-\sqrt{9-x^2}}{x} - \arcsin\left(\tfrac{x}{3}\right) + C$$



$\cot(u) = \frac{A}{O} = \frac{\sqrt{9-x^2}}{x}$

Ex: $\int \frac{dt}{\sqrt{t^2+4}}$

$t = 2\tan u$

$\frac{t}{2} = \tan u$

$\arctan\left(\frac{t}{2}\right) = u$

$dt = 2\sec^2 u \, du$

$$= \int \frac{2\sec^2 u \, du}{\sqrt{4\tan^2 u + 4}}$$

$$= \int \frac{2\sec^2 u \, du}{2\sec u}$$

$$= \int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$= \ln\left|\sec\left(\arctan\left(\tfrac{t}{2}\right)\right) + \tan\left(\arctan\left(\tfrac{t}{2}\right)\right)\right| + C$$

$$= \ln \left| \frac{\sqrt{x^2+4}}{2} + \frac{x}{2} \right| + C$$