6.8 - Indeterminate forms and L'Hospital's Rule.

Consider

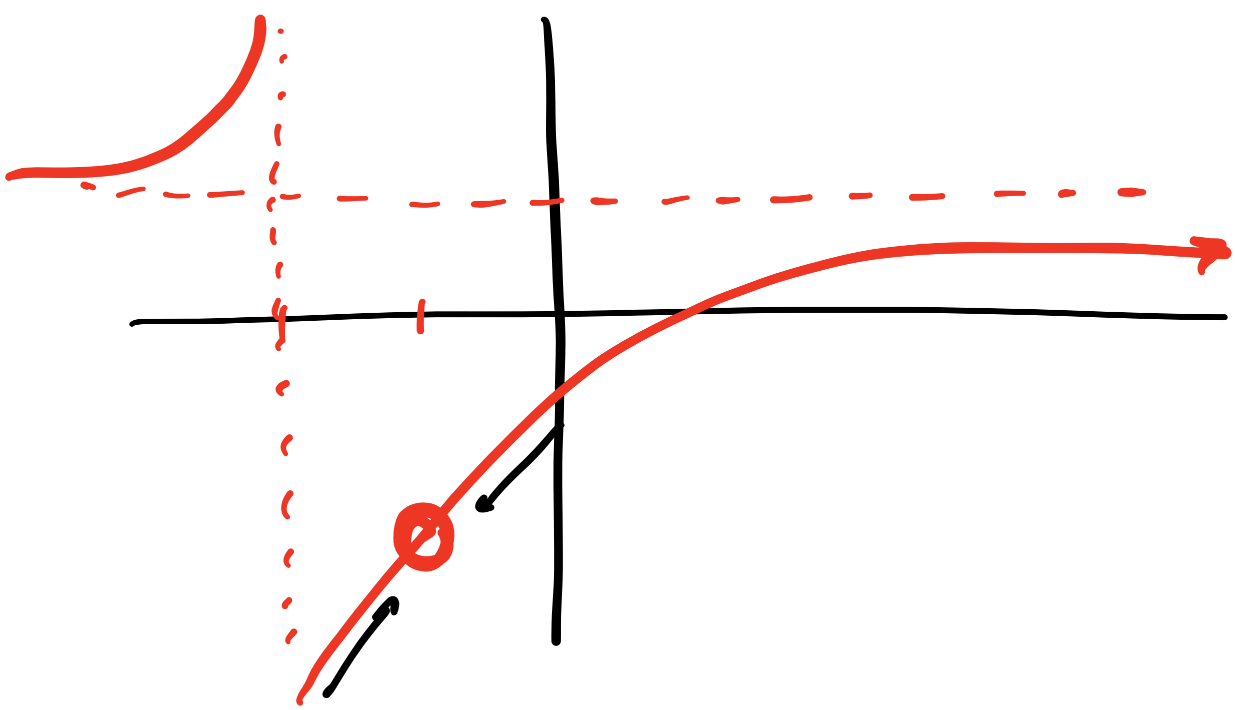
$$\lim_{x \to -1} \frac{x^2-1}{x^2+3x+2}.$$

we can't plug in, because $(-1)^2-1=0$ and $(-1)^2+3(-1)+2=0$

So the limit makes no sense.

$\frac{0}{c} \to 0$ $\quad \frac{c}{0} \to \pm\infty$. but $\frac{0}{0}$? we just don't know.

One strategy is to check the graph.



at $x=-1$, the graph has a hole in it.
But the limit here still exists!

from the graph, it seems that

$$\lim_{x\to -1} \frac{x^2-1}{x^2+3x+2} = -2.$$

can we verify? let's use algebra to eliminate this removeable singularity

$$\lim_{x\to -1} \frac{x^2-1}{x^2+3x+2} = \lim_{x\to -1} \frac{(x+1)(x-1)}{(x+2)(x+1)} = \lim_{x\to -1} \frac{x-1}{x+2} = \frac{-2}{1} = -2.$$

This generally works for rational functions.

if $\lim_{x\to a} p(x)=0$, $\lim_{x\to a} q(x)=0$,

then we can factor out a common factor and simplify to solve $\lim_{x\to a} \frac{p(x)}{q(x)}$

but what about something like

$$\lim_{x\to 1} \frac{\ln x}{x-1} ? \qquad \begin{matrix} \ln x \to 0 \\ x-1 \to 0 \end{matrix} \text{ as } x\to 1$$

but we can't factor $\ln x$.

__Theorem:__ Suppose f, g are differentiable near $x=a$,

$\lim_{x\to a} f(x) = 0$ or $\pm\infty$

$\lim_{x\to a} g(x) = 0$ or $\pm\infty$.

then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

why? let's look at the special case $f(a) = g(a) = 0$.
and f, g diff at a.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} \overset{L'H}{=} \lim_{x\to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1} \frac{1/x}{1} = \boxed{1}.$$

<u>other forms:</u>

<u>product:</u> $f(x)\,g(x)$ $\quad\quad 0\cdot\infty.$

$$fg = \frac{f}{\frac{1}{g}}.$$

$$\lim_{x\to 0^+} x\ln(x)$$

$x\to 0 \quad \ln(x)\to -\infty$

$0\cdot(-\infty)$ undeterminate

$$= \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \quad \frac{-\infty}{\infty}$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0} -x = \boxed{0}$$

power form:

$\lim_{x\to a} f(x)^{g(x)}$ is indeterminate if

$$0^0 \quad , \quad 1^\infty \quad , \quad \infty^0$$

$$f^g = e^{\ln f^g} = e^{g \ln f} = e^{\frac{\ln f}{1/g}}$$

$$\lim_{x\to 0^+} x^x$$

$$= \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x \ln x} = \lim_{x\to 0^+} e^{\frac{\ln x}{1/x}}$$

$$= e^{\boxed{\lim_{x\to 0^+} \frac{\ln(x)}{1/x}}}$$

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{LH}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x$$

$= 0$

$$= e^0$$

$$= 1.$$

Might need to repeat

$$\lim_{x\to+\infty} \frac{e^x}{x^2} \overset{LH}{=} \lim_{x\to+\infty} \frac{e^x}{2x} = \lim_{x\to+\infty} \frac{e^x}{2} = \infty$$

$$\frac{\infty}{\infty} \qquad \frac{\infty}{\infty} \qquad \frac{\infty}{2}$$

Difference Form: $\infty - \infty$.

$$\lim_{x\to\frac{\pi}{2}^-} \sec x - \tan x$$

$$\infty - \infty$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x} \qquad \frac{0}{0}$$

$$\overset{LH}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = \frac{0}{1} = 0.$$