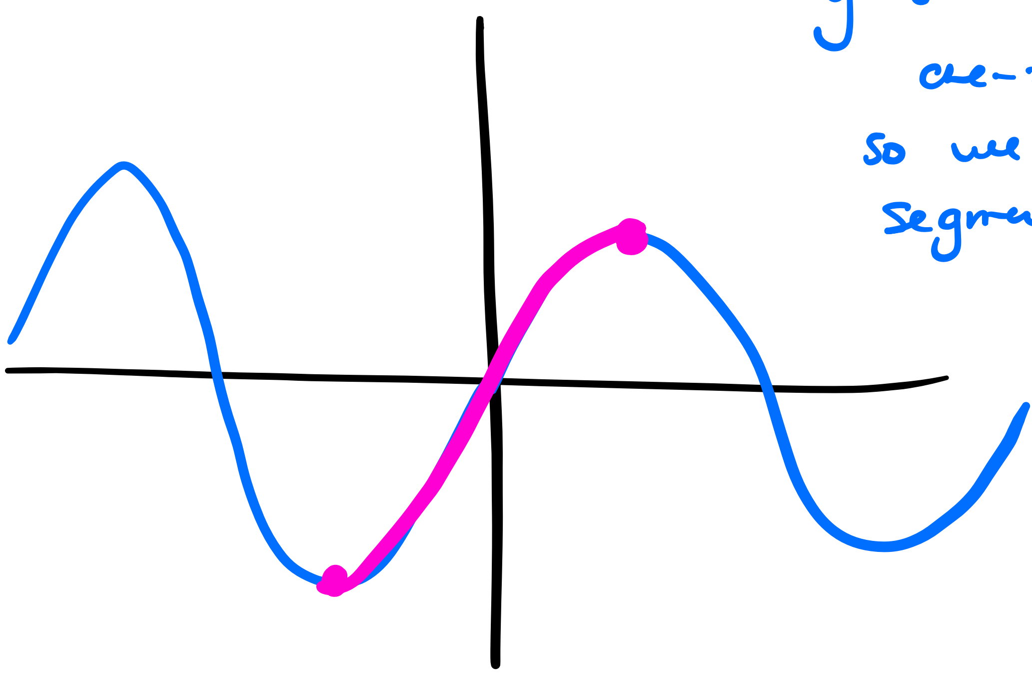
Inverse Trig Functions

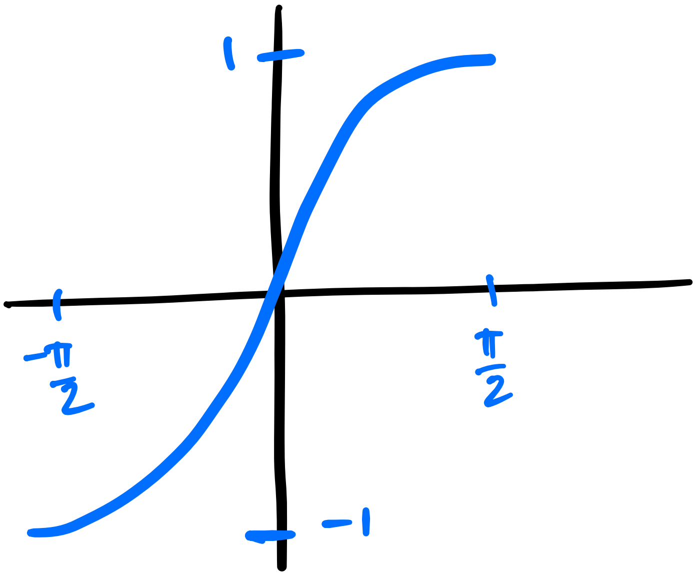
The most important examples in calculus involve trig functions. Let's go about discovering their inverse functions and properties.

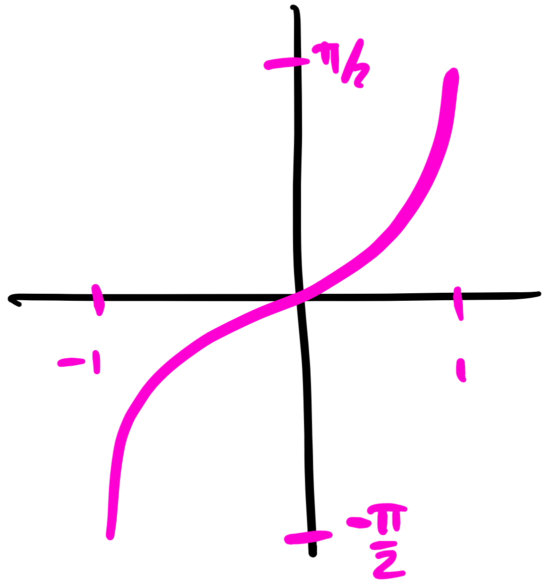


$y = \sin x$ is <u>not</u> one-to-one.
So we choose a segment that is

$y = \sin x, \quad D = [-\frac{\pi}{2}, \frac{\pi}{2}]$
$R = [-1, 1].$

Since this is increasing, it must have an inverse.





$y = \sin x$

$D: [-\frac{\pi}{2}, \frac{\pi}{2}]$

$R = [-1, 1]$

input: angle

output: y-coordinate on unit circle

$y = \sin^{-1}(x) \neq \frac{1}{\sin x}$!

$= \arcsin(x)$

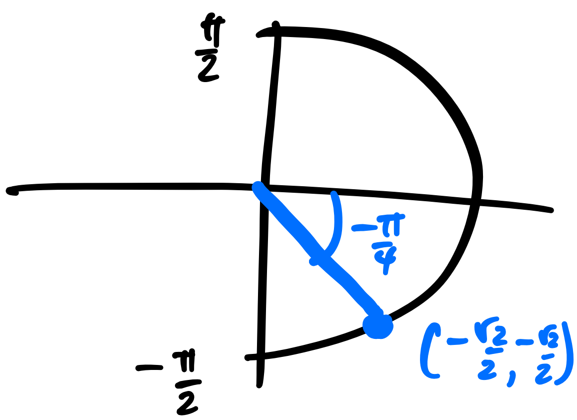
$D = [-1, 1]$

$R = [-\frac{\pi}{2}, \frac{\pi}{2}]$

input: y-coord

output: angle.

Ex: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \text{some angle} = -\frac{\pi}{4}$



Cancellation rules:

$\sin(\sin^{-1}(x)) = x \qquad x \text{ in } [-1,1].$

$\sin^{-1}(\sin(x)) = x \qquad x \text{ in } [-\frac{\pi}{2}, \frac{\pi}{2}].$

Derivative of $\sin^{-1}(x)$?
(might surprise you!)

$\sin^{-1}(x) = y \quad \text{when} \quad \sin(y) = x.$

$$\frac{d}{dx}\sin(y) = \frac{d}{dx}(x)$$

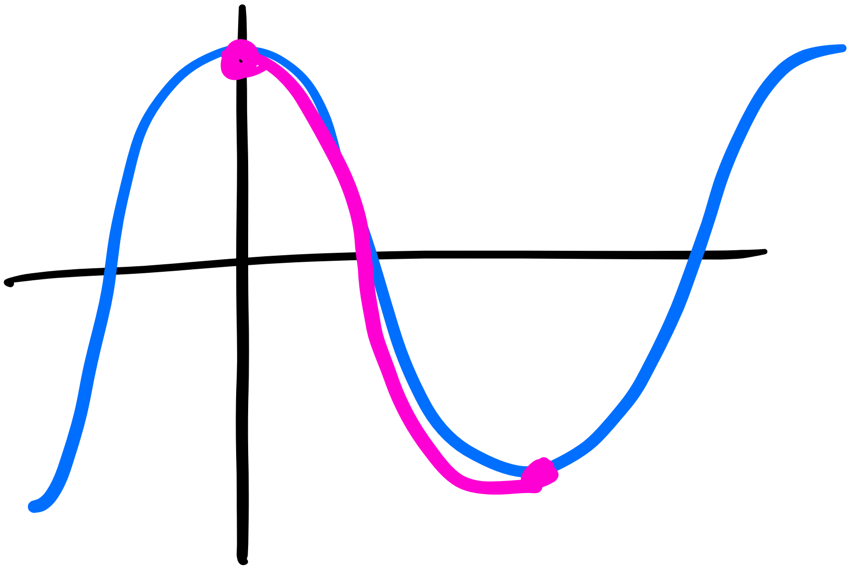
implicit derivative

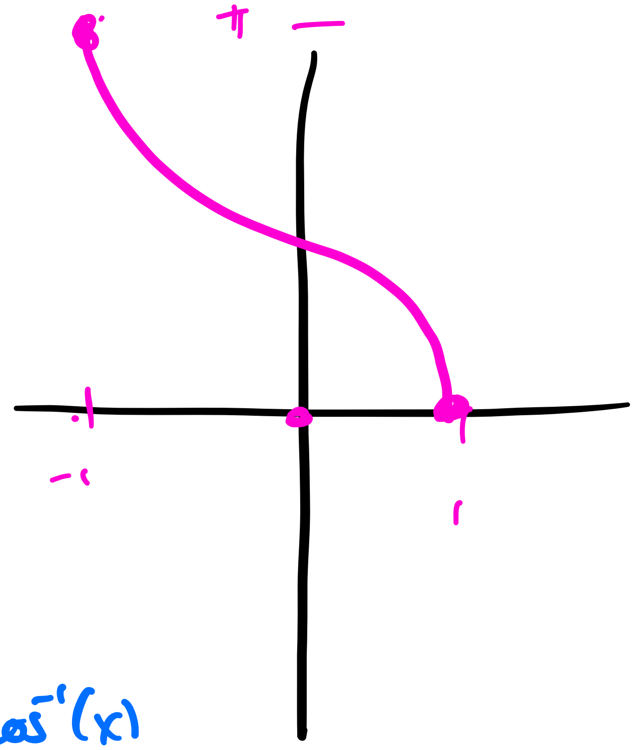
$$\cos(y)\cdot y' = 1$$

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}!$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

how about $y = \cos^{-1}(x)$?





$y = \cos x$

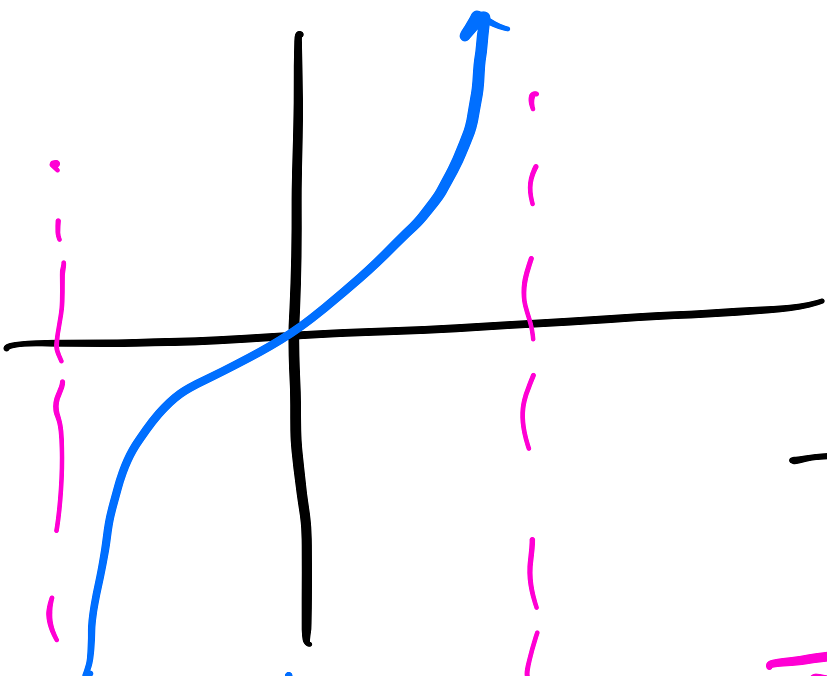
$D = [0, \pi]$

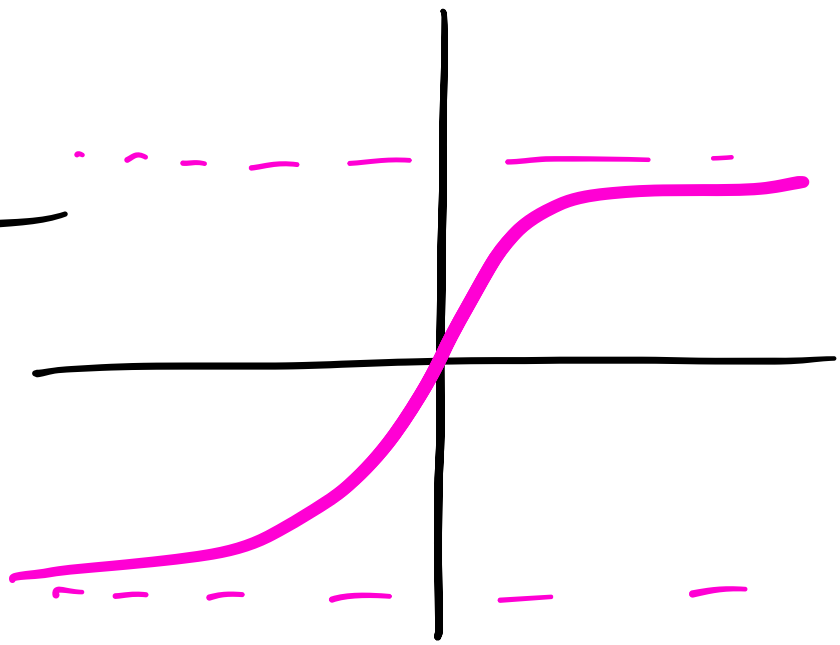
$R = [-1, 1]$

$y = \cos^{-1}(x)$
$= \arccos(x)$

$D = [-1, 1]$

$R = [0, \pi]$.





$y = \tan x$
$D = [-\frac{\pi}{2}, \frac{\pi}{2}]$
$R = (-\infty, \infty)$

$y = \arctan x = \tan^{-1}(x)$.

$y = \tan^{-1}(x)$ if $\underline{\underline{x = \tan(y)}}$.

$$\frac{d}{dx} x = \frac{d}{dx} \tan(y)$$

$$1 = \sec^2(y) \cdot y'$$

$$y' = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2+1} \, !$$

$$\frac{d}{dx} \arctan(x) = \frac{1}{x^2+1}.$$

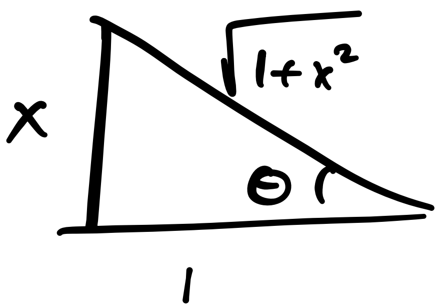
$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$
$\frac{d}{dx} \arctan x = \frac{1}{x^2+1}$	$\int \frac{1}{x^2+1} dx = \arctan x + C.$

Pythagorean evaluations

evaluate $\cos(\underbrace{\arctan x}_{\text{angle}})$.

$$\arctan x = \theta$$

$$\frac{O}{A} \qquad \frac{x}{1} = \tan\theta$$



$$\cos(\theta) = \frac{1}{\sqrt{1+x^2}} = \cos(\underbrace{\arctan x}_{\text{angle}})$$

Examples:

① $\sin(\tan^{-1}x)$

② $\frac{d}{dx}\sin^{-1}(2x+1)$

③ $\int \frac{1}{x^2+4}\,dx$

④ $\int \frac{t^2}{1+t^6}\,dt.$