The general exponential function

Consider $f(x) = 2^x$.

If x is an integer, this is easy to define. $f(3) = 2^3$.

If x is a rational number $\frac{p}{q}$,

we can still easily write

$$f\left(\frac{p}{q}\right) = 2^{p/q} = \sqrt[q]{2^p}.$$

but what about $x = \pi$? this is irrational.

Define: $b^x = e^{x \ln b}$ (everything here makes sense)

$$b^x = e^{\ln b^x} = e^{x \ln b}.$$

This is called the general exponential function.

properties:

① $b^{x+y} = b^x b^y$

② $b^{x-y} = b^x / b^y$

③ $(b^x)^r = b^{xr}$

$$\frac{d}{dx} b^x = \frac{d}{dx} e^{x\ln b} = e^{x\ln b} \cdot \frac{d}{dx}(x\ln b)$$

$$= e^{x\ln b}(\ln b)$$

$$= b^x \cdot \ln b$$

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integration If $\frac{d}{dx} b^x = b^x \cdot \ln b$,

$$\int b^x \ln b \, dx = b^x + C$$

so $\int b^x dx = \frac{1}{\ln b} b^x + C$

Ex: $\int_2^5 2^x dx = \ln 2 \cdot 2^x \Big|_2^5$

$= \ln 2(2^5 - 2^2).$

Types of power derivatives:

① $\frac{d}{dx}(b^n) = 0$ $\qquad$ $\frac{d}{dx}(2^3) = 0$

② $\frac{d}{dx}(x^n) = nx^{n-1}$ $\qquad$ $\frac{d}{dx}(x^{3/2}) = \frac{3}{2}x^{1/2}$

③ $\frac{d}{dx}(a^x) = \ln a \cdot a^x$ $\qquad$ $\frac{d}{dx}(3^x) = \ln 3 \cdot 3^x$

④ $\frac{d}{dx}(f(x)^{g(x)}) = \frac{d}{dx}\left(e^{g(x)\ln(f(x))}\right)$

$$\frac{d}{dx}(x^x) = \frac{d}{dx}\left(e^{x\ln x}\right) = e^{x\ln x}(\ln x + 1)$$

$$= x^x(\ln(x)+1).$$

General Logarithms

for $b>0$ $b\neq 1$, b^x is one-to-one, and so has an inverse. we call it the logarithm base b and write $y=b^x \Longleftrightarrow x=\log_b y$.

$\log_e x = \ln(x)$.

$$\log_b(b^x) = x \qquad b^{\log_b x} = x.$$

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every log is a natural log in disguise:

$$\log_b x = y \Rightarrow b^y = x$$

$$\ln b^y = \ln x$$

$$y \ln b = \ln x$$

$$y = \frac{\ln x}{\ln b}.$$

so $\log_b x = \dfrac{\ln x}{\ln b}$

$$\frac{d}{dx}\left(\log_b x\right) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{x \ln b}.$$