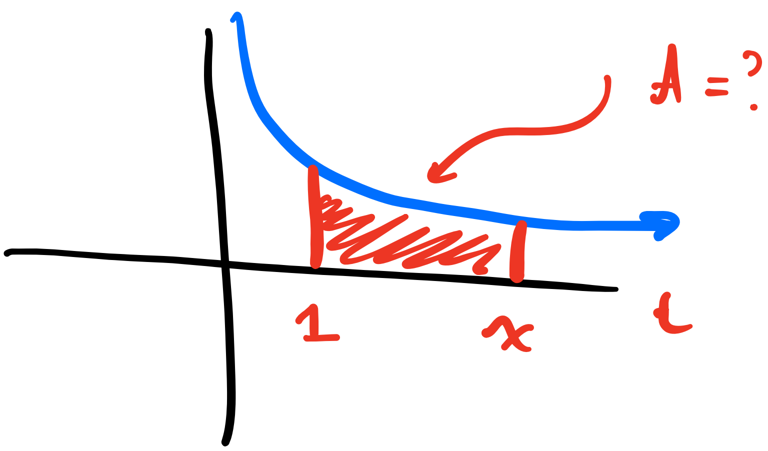
The natural logarithm

A function that shows up all the time in mathematical modeling and applications is $f(x) = 1/x$.

A typical question to ask is "what is the area under $1/x$?"



This obviously exists. Symbolically,

$$A = \int_1^x \frac{1}{t}\,dt$$

Let's give this function a symbol

$$F(x) = \int_1^x \frac{1}{t}\,dt$$

FTC: $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

so $F(x)$ is a function so $F'(x) = \frac{1}{x}.$

$$F(x) = \int_1^x \frac{1}{t}\,dt$$

$$F'(x) = 1/x$$

$F(1) = 0$ since $\int_1^1 f(t)\,dt = 0.$

We call $F(x)$ the natural logarithm

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$

$$\frac{d}{dx}\ln(x) = 1/x$$

$$\ln(1) = 0$$

other properties?

1. let $f(x) = \ln(ax)$

$$f'(x) = \frac{1}{ax} \cdot a = \frac{1}{x}$$

so $\ln(ax)$ and $\ln(x)$ have the same derivative,

hence $\ln(ax) = \ln(x) + C$

plug in $x = 1$

$$\ln(a) = \ln(1) + C$$

$$\underline{\ln(a) = C}$$

So $\ln(ax) = \ln(a) + \ln(x)$

2. $\ln\left(y \cdot \frac{1}{y}\right) = \ln(y) + \ln\left(\frac{1}{y}\right)$

$\quad \shortparallel$

$\ln(1)$

$\quad \shortparallel$

$\quad 0$

so $\ln\left(\frac{1}{y}\right) = -\ln(y)$

hence $\ln\left(\frac{x}{y}\right) = \ln\left(x \cdot \frac{1}{y}\right) = \ln(x) + \ln\left(\frac{1}{y}\right)$

$= \ln(x) - \ln(y).$

3. $\ln(x^r) = r\ln(x)$

why?

$$\frac{d}{dx}\ln(x^r) = \frac{1}{x^r}\cdot x^{r-1}\cdot r = \frac{r}{x}.$$

also $$\frac{d}{dx} r\ln(x) = \frac{r}{x}.$$

so $$\ln(x^r) = r\ln(x) + C$$

so $C = 0$.

Consequences

real power rule

if $n \neq 1$, $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1}$

if $n = -1$, $\displaystyle\int x^{-1}\,dx = \ln|x| + C.$

Examples:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

let $u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$$= \int \frac{-du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

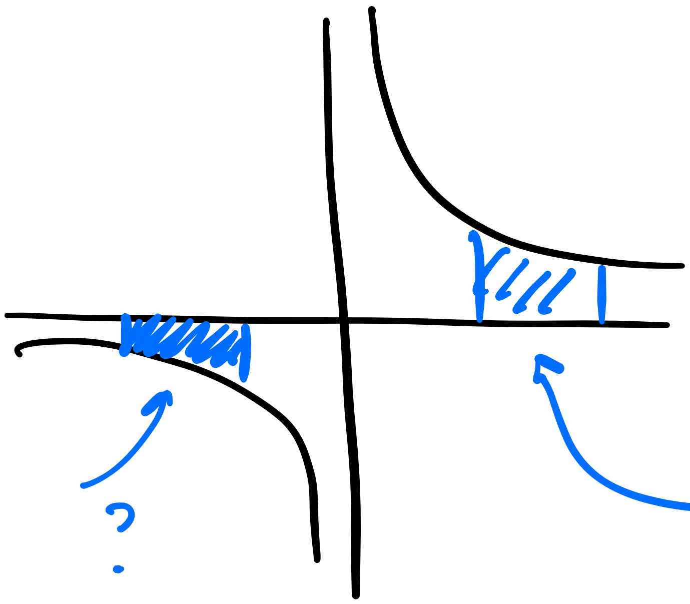
$$= \ln|\sec x| + C.$$

$$\frac{d}{dx} \ln\left(\frac{(x+1)(x-1)}{(x^2+1)}\right)$$

$$= \frac{d}{dx}\left(\ln(x+1) + \ln(x-1) - \ln(x^2+1)\right)$$

$$= \frac{1}{x+1} + \frac{1}{x-1} - \frac{1}{x^2+1}(2x).$$

why absolute value?



notice this area is equal to this

$$\int_{-x}^{-1} \frac{1}{t}\,dt \qquad \text{let } \boxed{u=-t}$$

$$du = -dt$$

$$u(-x) = -(-x) = x$$

$$u(-1) = -(-1) = 1$$

$$= \int_1^x \frac{1}{u}\,du.$$

$$= \ln(x)$$

$$= \underline{\ln|-x|}$$

so $\int \frac{1}{x}dx = \ln x$ if $x>0$,

$\int \frac{1}{x}dx = \ln(-x)$ if $x<0$

so $\int \frac{1}{x}\,dx = \ln|x| + C$.