We're going to study functions and their inverses.

We use inverses all the time - an inverse of a function "undoes" the function

---

So what is a function?

- f is a <u>rule</u> to take input x to an output y (sometimes called $f(x)$)
- D is a <u>domain</u> of allowable inputs
- R is the set of outputs for x in D.

we write $f: D \to R$

Informally, we can think of an inverse as "reversing" f.

If $f(x) = y$ then $f^{-1}(y) = x$.

(x, y) on graph of f

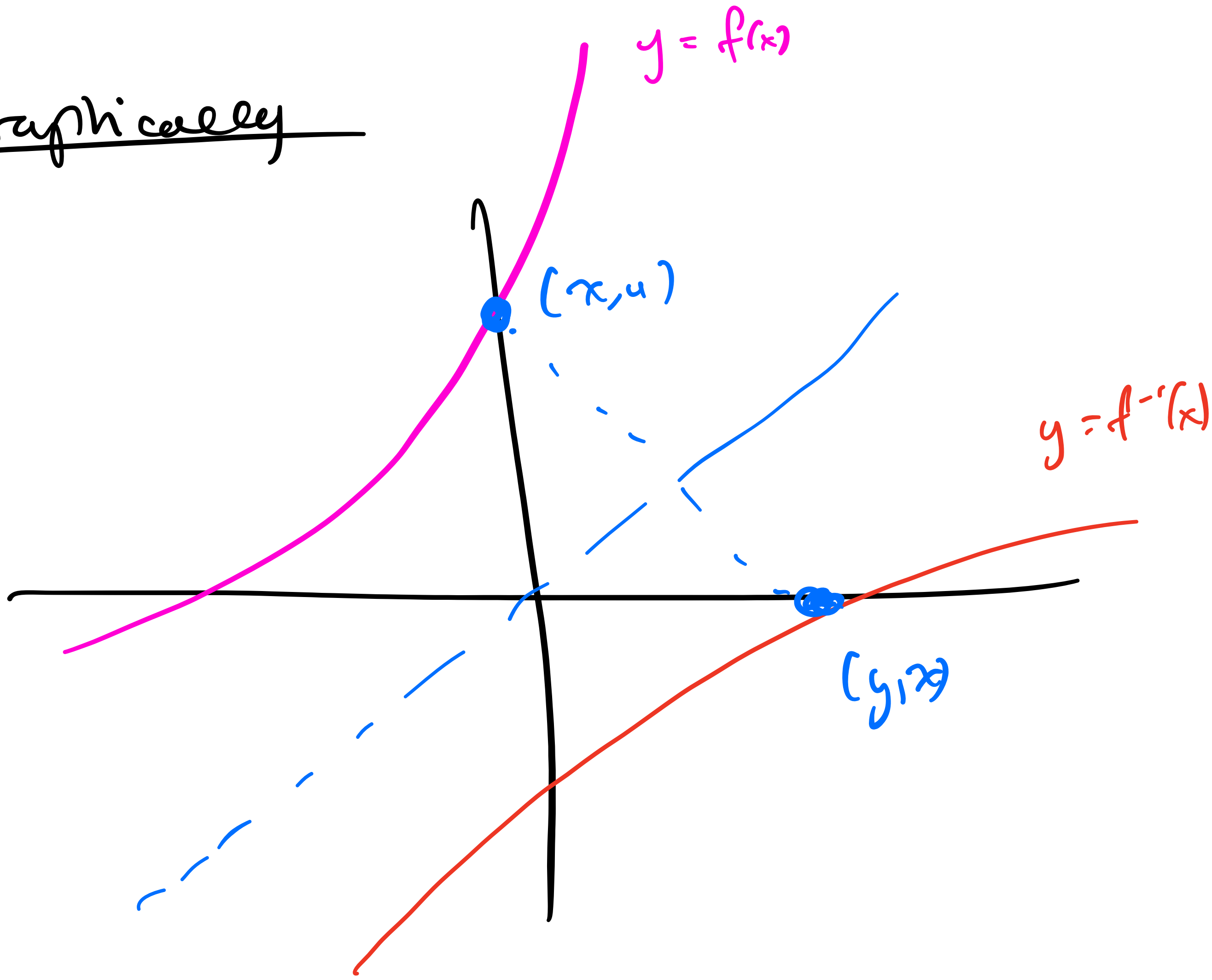
(y, x) on graph of f^{-1}

Formally

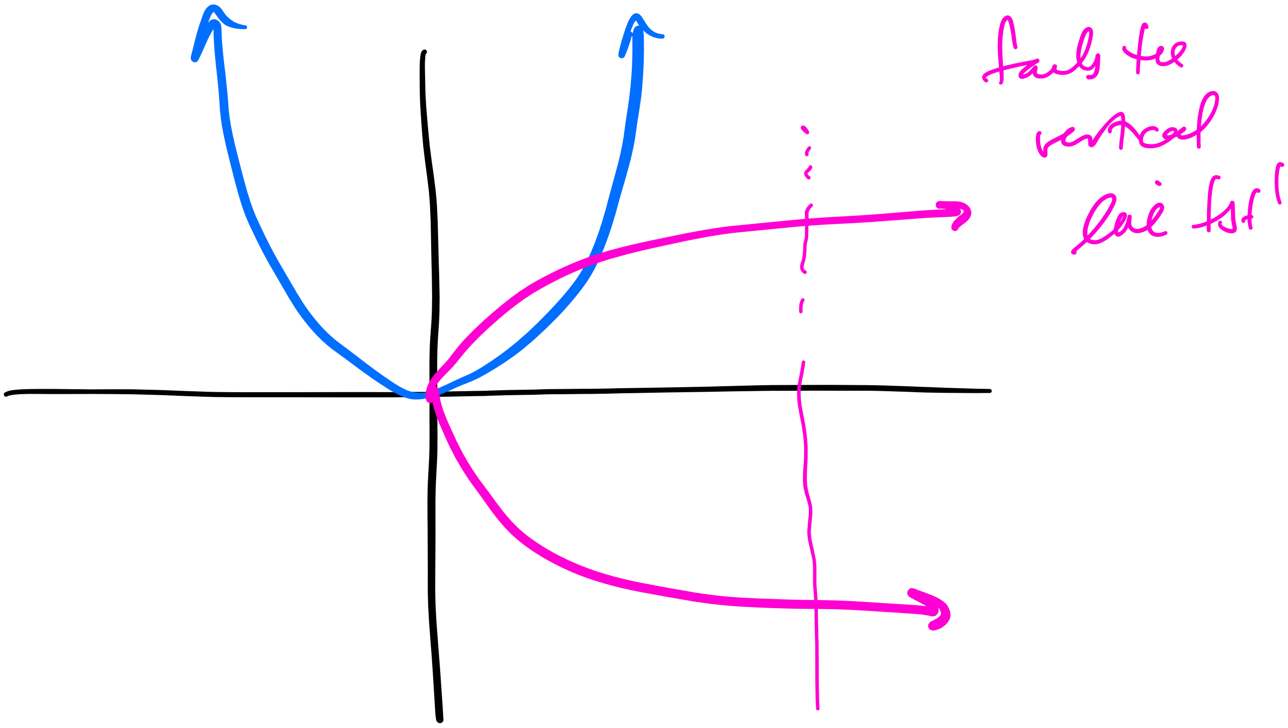
The inverse of $f: D \to R$ is a function $f^{-1}: R \to D$ so that

$$f^{-1}(y) = x \quad \text{whenever} \quad y = f(x).$$

Graphically

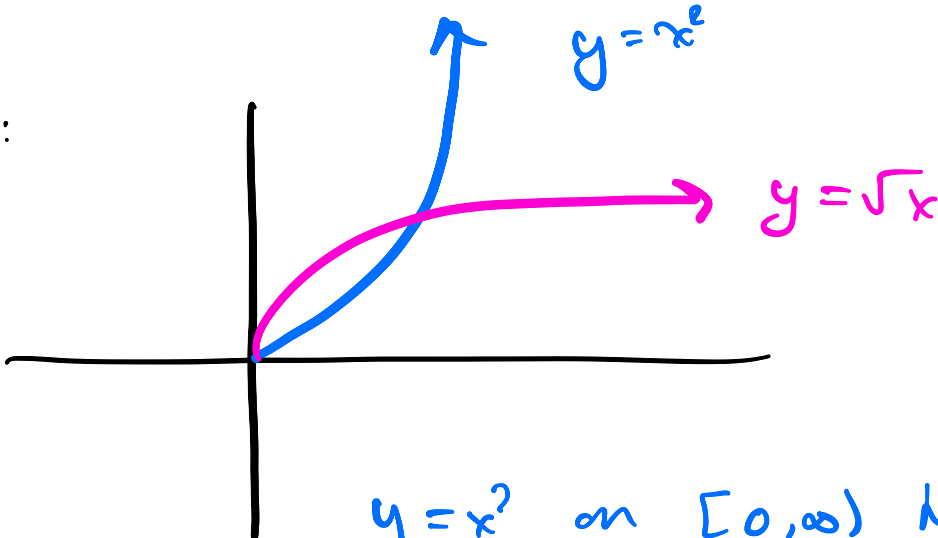


One big demand: the inverse of a function should be a function!

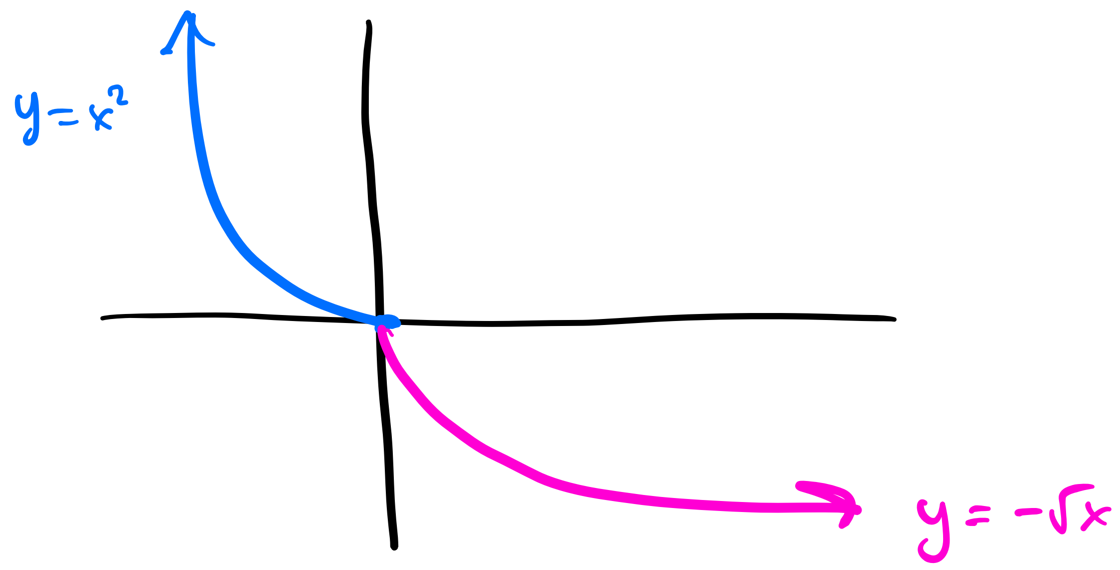


$y = x^2$ on $(-\infty, \infty)$ doesn't have an inverse!

instead:



$y = x^2$ on $[0, \infty)$ has inverse $y = \sqrt{x}$



$y=x^2$ on $(-\infty,0]$ has inverse $y=-\sqrt{x}$.

If y is in the range of f implies that only one x in the domain has $f(x)=y$, then f is called a <u>one-to-one</u> function.

<u>OR</u>

$$f(x) \neq f(y) \Rightarrow x \neq y \quad \text{for } x,y \in D.$$

<u>OR</u>

If x,y in D and $f(x)=f(y)$, then $x=y$.

Definition Let f be one-to-one on an interval (a,b) with domain $A \subseteq (a,b)$ and range B.

Then f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \text{ whenever } f(x) = y.$$

How can we find an inverse?

(1) To show f, g are inverses, show that $f(g(x)) = x$ (f undoes g)

(2) To find f^{-1} given f

(A) write $y = f(x)$

(B) solve for x in terms of y

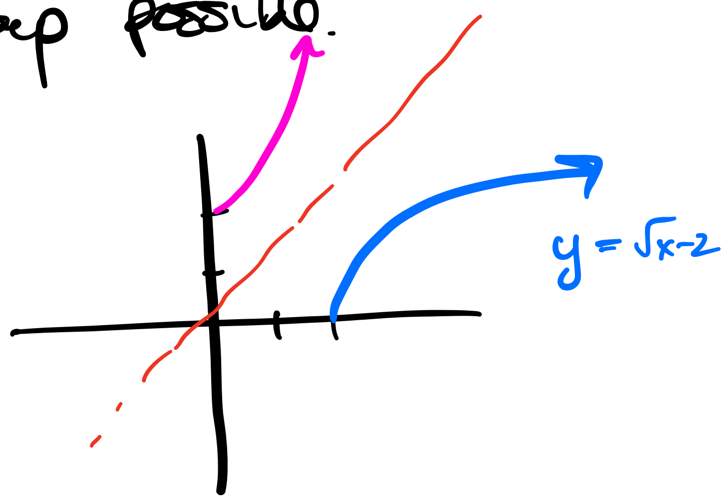
(C) switch x and y to get $y = f^{-1}(x)$.

Note: this isn't always possible.

Example: $f(x)=\sqrt{x-2}$

D: $[2,\infty)$

R: $[0,\infty)$



(A) $y=\sqrt{x-2}$

(B) $y^2=x-2$

$y^2+2=x$

(C) $y=x^2+2$

D: $[0,\infty)$

R: $[2,\infty)$

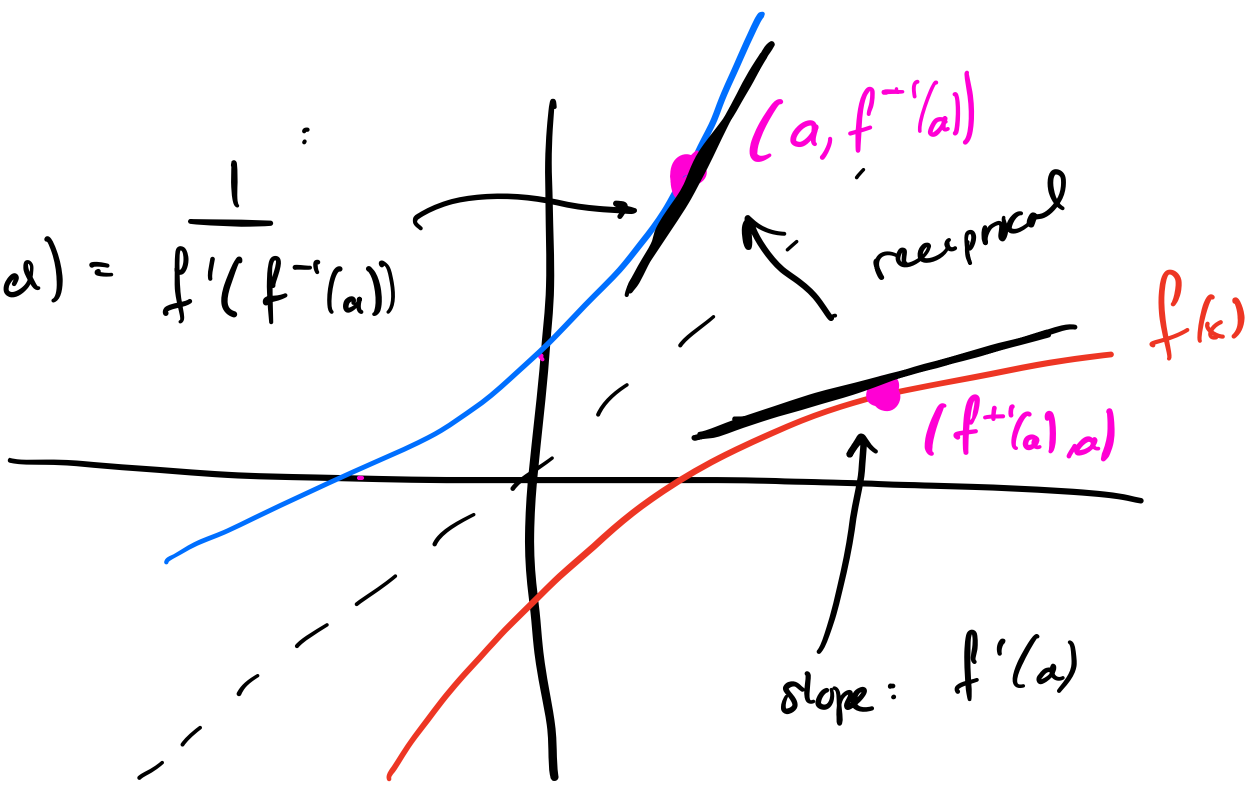
Calculus and inverses

Thm: If f is continuous on (a,b) and f is one-to-one, then f^{-1} is continuous on $f(a,b)$ = range of f on (a,b).

Thm: If f is one-to-one and differentiable then f^{-1} is also one-to-one and differentiable with

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} \quad \text{if } f'(f^{-1}(a)) \neq 0$$

$$f^{-1\,\prime}(a) = \frac{1}{f'(f^{-1}(a))}$$



why? $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$

call $f^{-1}(a) = b$ call $f^{-1}(x) = y$

so $f(b) = a$ so $f(y) = x$.

then $$(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$$

if $x \to a$,

$f(y) \to f(b)$,

so $y \to b$ (since f is cts and 1-1)

$$= \lim_{y \to b} \frac{y - b}{f(y) - f(b)}$$

$$= \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}$$

Examples:

① find f^{-1} of $f(x) = x^3+1$

② show $f = x^3+1$, $g = \sqrt[3]{x-1}$ are inverses

③ if $f(x) = 2x + \cos x$, find $(f^{-1})'(1)$

(information about f^{-1} w/o computing f^{-1}.)

$y = 2x + \cos x$ cannot be solved for x.

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

$1 = f(x) = 2x + \cos x$

$x = 0$ (guess we got lucky)

so $(f^{-1})'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$

$f'(x) = 2 - \sin x$

$f'(0) = 2$