Enumeration Functions for Formal Languages

ILAS Session JMM 2026

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joint with J. E. Pascoe
(Drexel)

A $\underline{\text{formal}}\ \overset{\mathcal{L}}{\underline{\text{language}}}$ is a set of $\underset{w}{\underline{\text{words}}}$ in an $\underset{\mathcal{A}}{\underline{\text{alphabet}}}$ of symbols.

(Silly)
Example: $\mathcal{A} = \{z\}$,

$\mathcal{L}$ = all words in z

$= \{1, z, z^2, z^3, \dots\}$.

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Enumeration Functions

The function $f(z) = 1 + z + z^2 + z^3 + \dots$

has every word in $\mathcal{L}$ as a monomial term.

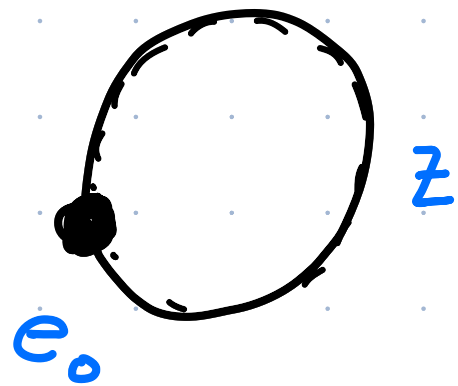
$$f(z) = \sum_{w \in \mathcal{L}} w = 1 + z + z^2 + z^3 + \dots$$

enumerates $\mathcal{L}$. or

f is the enumeration function for $\mathcal{L}$.

Graphical View

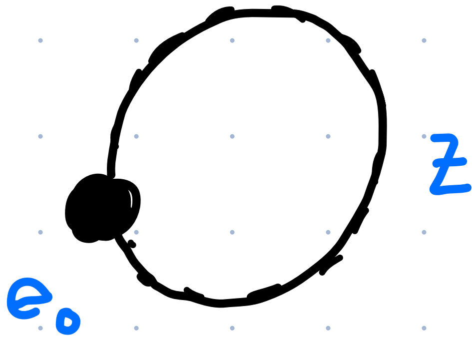
Consider walks on the weighted graph



walk length:	weight	
0	1	e_0
1	z	$e_0 \xrightarrow[z]{} e_0$
2	z^2	$e_0 \xrightarrow[z]{} e_0 \xrightarrow[z]{} e_0$

walk generating function: $1 + z + z^2 + z^3 + \dots$

graph | walk generating function



$$\Rightarrow \quad f(z) = 1 + z + z^2 + z^3 + \ldots$$

also

$$f(z) = \sum_{w \in \mathcal{J}} w \quad \text{enumerates}$$

$$\mathcal{J} = \{1, z, z^2, \ldots\}$$

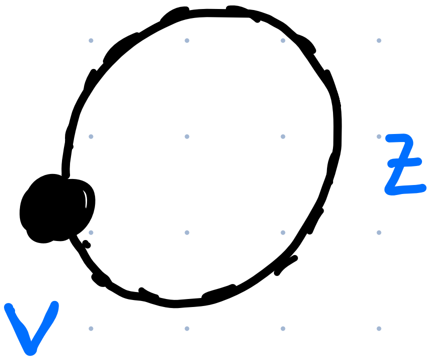
$\boxed{\text{and}}$

$$(1-z)^{-1} = f(z) = \sum_{w \in \mathcal{J}} w$$

"realization" for f

enumeration function for $\mathcal{J}$.

graph



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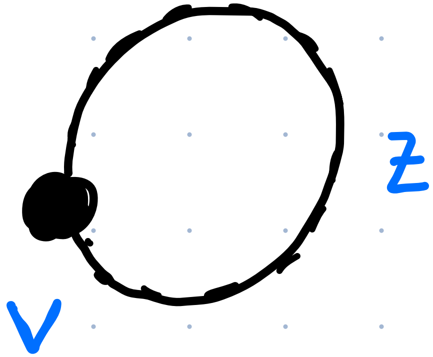

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also

$f(z) = \sum_{w \in \mathcal{L}} w$ enumerates

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Plan For Talk

- Develop an interesting example
  Dyck language
- Introduce graph structure
- Use matrix positivity to get a realization
- State a result about numerical radius

A More Interesting Example

Consider the alphabet

$$\mathcal{A} = \{ [,] \},$$

and $\mathcal{D}$ the set of <u>balanced</u> <u>brackets</u>

$[\,[\,]\,] \in \mathcal{D}$

$[\,[\,] \notin \mathcal{D}$ unequal #s.

$[\,]\,][\notin \mathcal{D}$ too many] before [.

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DYCK LANGUAGE

$\mathcal{D}$ is called the Dyck language.

$\mathcal{D}$ has a natural involution:

$$[^* =].$$

Let $z = [$, $w =]$.

and $z^* = w$.

then $([[][[]]])^*$

$= (zzwzzwww)^* = zzzwwzww.$

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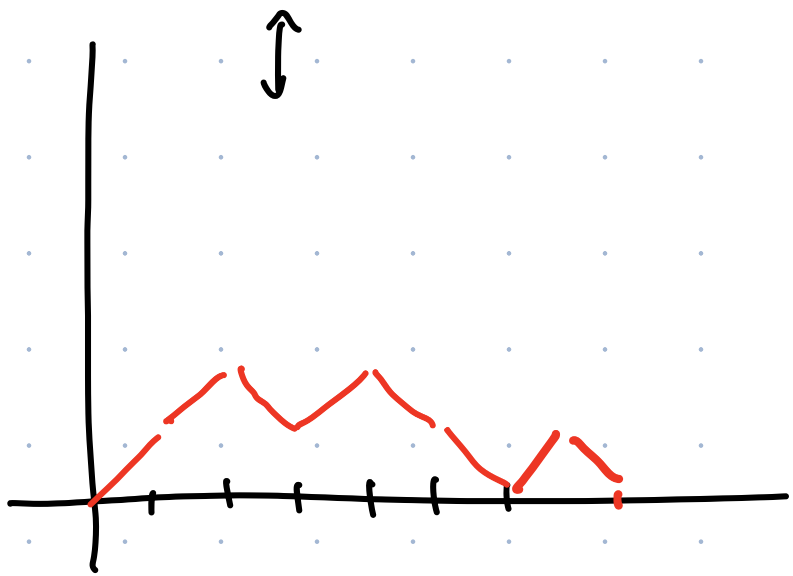

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- recursive construction

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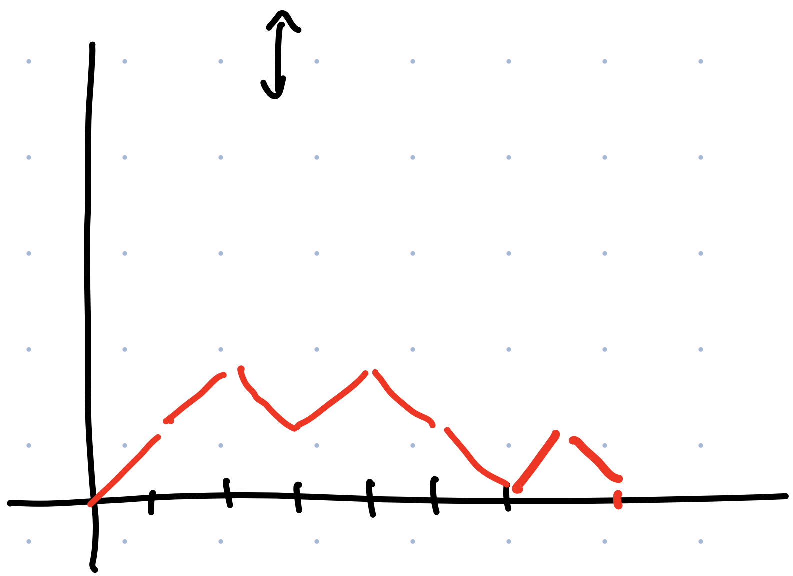

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STRUCTURE OF DYCK WORDS

If $w_1 = zzwzzwww \in \mathcal{D}$,

then $w_2 = w_1^* = zzzwwzww \in \mathcal{D}$.

$\mathcal{D}$ is closed under involution.

A formal language $\mathcal{L}$ is called self-adjoint if $\mathcal{L}$ is closed under involution

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Structure of Dyck words

$$\omega_1 = \underbrace{z\,w\,z\,z}_{\beta^*}\,\underbrace{w\,w}_{\alpha} \qquad\qquad \omega_2 = \underbrace{z\,z\,z\,w}_{\gamma^*}\,\underbrace{w\,w}_{\alpha}$$

Note: $\alpha^*\alpha = (ww)^* ww = zzww \in \mathcal{D}$.

$$\gamma^*\beta = zzzw(zwzz)^*$$
$$= zzzwwwzw \in \mathcal{D}$$

A self-adjoint language $\mathcal{J}$ is Pythagorean if $\gamma^*\alpha \in \mathcal{J}$ and $\beta^*\alpha \in \mathcal{J}$ imply $\alpha^*\alpha \in \mathcal{J}$ and $\gamma^*\beta \in \mathcal{J}$.

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Completable Words

$\alpha = ww$

$\beta = wwzw$

$\gamma = zwww$

These words can be completed to form a word in $\mathcal{D}$.
$(\beta^*\alpha,\ \gamma^*\alpha,\ \beta^*\gamma \in \mathcal{D})$

Let $\mathcal{G}$ be the set of all words in $\mathcal{A}$.

the set of completable words (ellipses) for $\mathcal{J} \subseteq \mathcal{G}$ is

$$E_{\mathcal{J}} = \{\alpha \in \mathcal{G} : \exists \beta \in \mathcal{G} \ni \beta^*\alpha \in \mathcal{J}\}.$$

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EQUIVALENCE IN $E_{\mathcal{J}}$

DEF: $\alpha, \beta \in E_{\mathcal{J}}$,

$\alpha \cong \beta$ if $\beta^{\dagger}\alpha \in \mathcal{J}$.

THM: If $\mathcal{J}$ is Pythagorean,

$\alpha \cong \beta$ is an equivalence relation

Graph Structure on $E_{\mathcal{D}}$.

$$E_{\mathcal{D}} = \{ e, w, zw, ww, zww, wzw, www, \ldots \}.$$

Draw a z-edge between α and β if $\beta^* z^* \alpha \in \mathcal{D}$.

For example,

$$\begin{tikzcd} zw \arrow[r, shift left, "z"] & wzw \arrow[l, shift left, "w"] \end{tikzcd}$$

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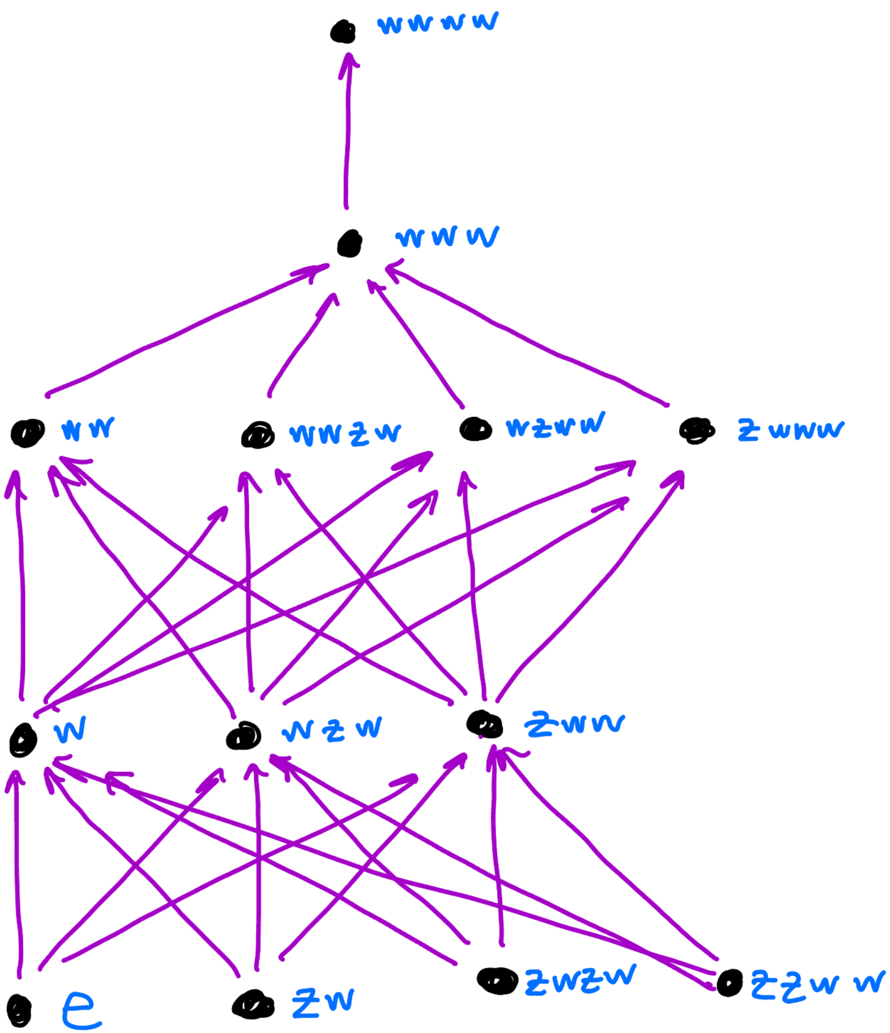

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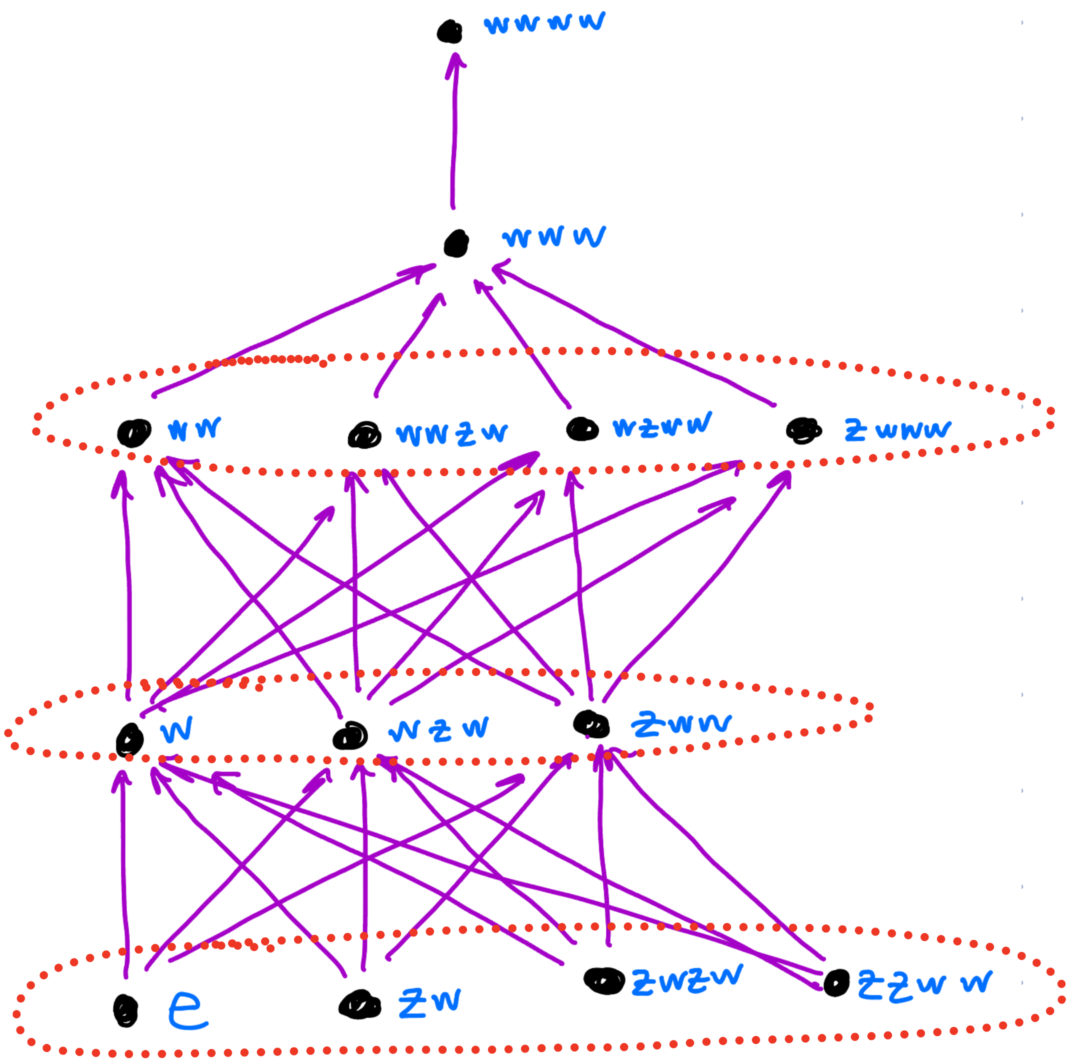

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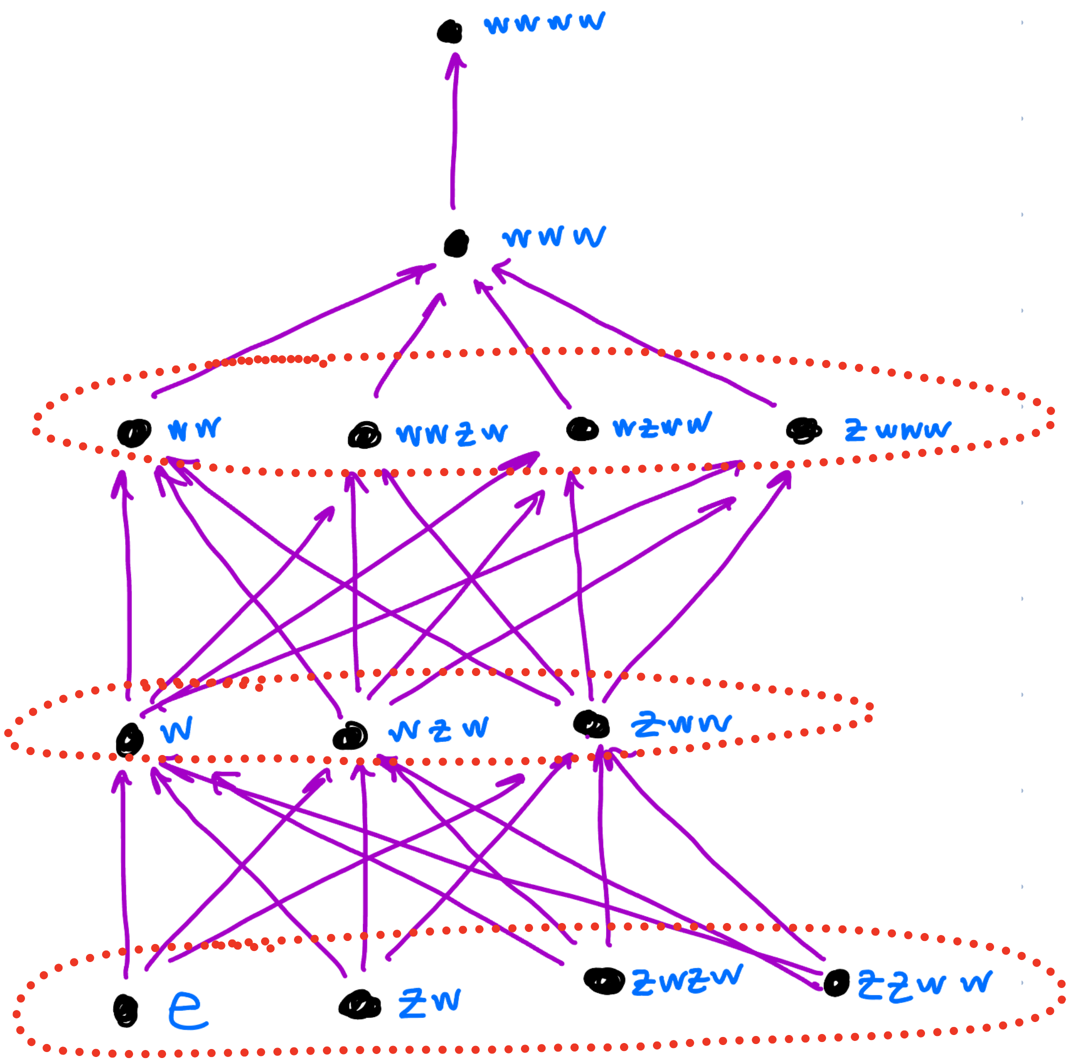
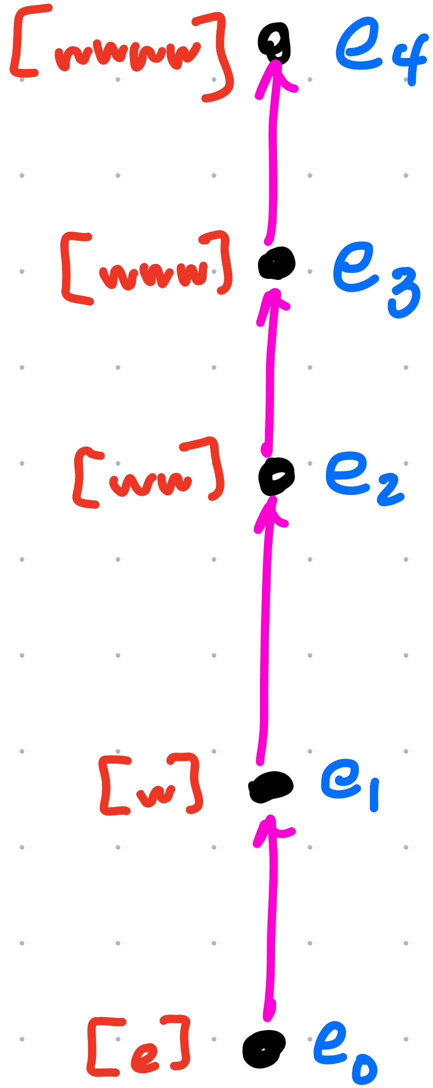
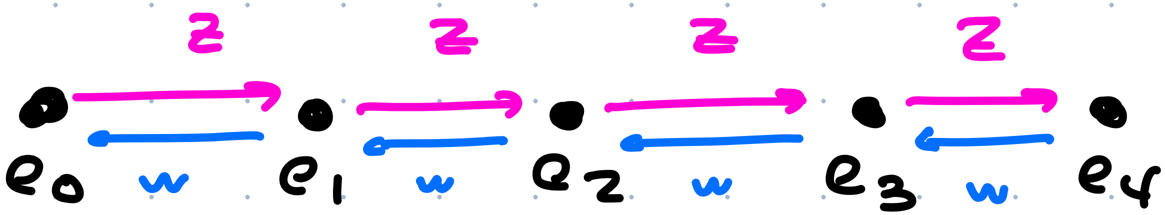




GRAPH WALKS AND ENUMERATION



Every walk beginning and ending at e_0 has weight $\omega \in \mathcal{D}$.

Example: $e_0 \xrightarrow[z]{} e_1 \xrightarrow[z]{} e_2 \xrightarrow[w]{} e_1 \xrightarrow[z]{} e_2 \xrightarrow[w]{} e_1 \xrightarrow[w]{} e_0$

So $f(z,w) = \sum_{\omega \in \mathcal{D}} \omega = e + zw + zzww + \dots$

is a walk-generating function

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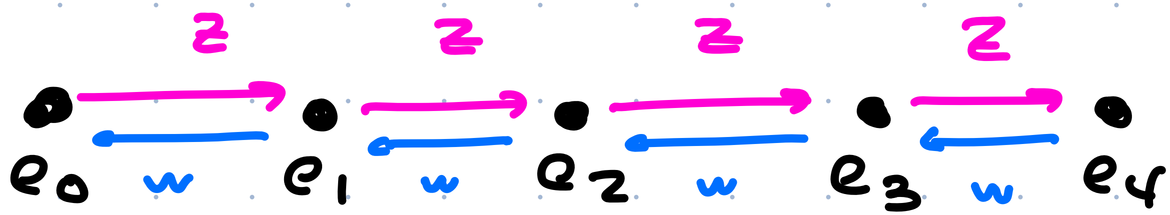

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Enumerating Matrix for $\mathcal{D}$.

$$C = \left[1_{\mathcal{D}}[\beta^* \alpha] \right]_{\alpha, \beta \in E_{\mathcal{D}}}.$$

$$
\begin{array}{cc|ccccc}
 & & & & \beta & & \\
 & & 1 & w & zw & zww & wzw \\
\hline
= & 1 & 1 & 0 & 1 & 0 & 0 \;\cdots \\
 & w & 0 & 1 & 0 & 1 & 1 \\
\alpha & zw & 1 & 0 & 1 & 0 & 0 \\
 & zww & 0 & 1 & 0 & 0 & 1
\end{array}
$$

POSITIVITY

$C = [1_{\mathcal{J}}(\beta^* \alpha)]_{\alpha,\beta}$ is a sort of Hankel matrix for $f = \sum_{\omega \in \mathcal{J}} \omega$.

C is self-adjoint since $\mathcal{J}$ is self-adjoint.

LEMMA: $C = [1_{\mathcal{J}}(\beta^* \alpha)]_{\alpha,\beta \in \mathcal{L}} \geq 0$ iff $\mathcal{J}$ is Pythagorean.

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Matrix Convexity

$$D = \{ A = (A_1 \ldots A_d) : \|A\| \le r ,\ A_i^* = A_i \}.$$ (convex free set)

DEF: Suppose $f : D \to$ selfadjoint. is an nc-function.

f is <u>matrix convex</u> if

$$f\left(\frac{A+B}{2}\right) \le \frac{f(A) + f(B)}{2} \quad \text{for all } A, B \in D.$$

<u>LEMMA:</u> f is matrix convex if and only if

$$C = [c_{\beta^t \alpha}]_{\alpha, \beta} \ge 0.$$

THM: Let $\mathcal{L}$ be a formal language with

$$f = \sum_{w \in \mathcal{L}} w.$$

<u>Then</u>

$\mathcal{L}$ is Pythagorean

iff

$$C = \left[1_{\mathcal{L}}(\beta^T \alpha) \right]_{\alpha, \beta} \geq 0$$

iff

$$f = \sum_{w \in \mathcal{L}} w \quad \text{is matrix convex.}$$

Roger Road

A BUTTERFLY REALIZATION

Let $\mathrm{irr}\,\mathcal{D}$ be the irreducible Dyck words.

Let $f(z,w) = \sum_{\omega \in \mathrm{irr}\,\mathcal{D}} \omega = 1 + zw + zzwwt\ldots$

Then

$$f(z,w) = (e_0^* \otimes 1)(I - S \otimes z - S^* \otimes w)^{-1}(e_0 \otimes 1)$$

where $S : e_k \mapsto e_{k+1}$

$S^* : e_{k+1} \mapsto e_k$ and $S^*(e_0) = 0.$

A Butterfly Realization

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\end{tikzcd}
$$

$\{e_k\}$ is a basis for $V = \mathcal{F}_D/\cong$

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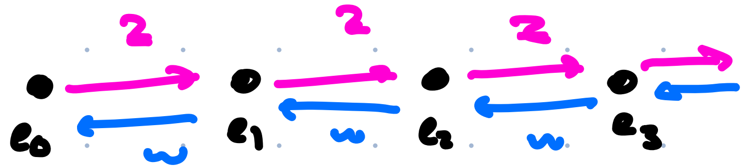
Summary



$$\Rightarrow \quad f(z,w) = 1 + zw + zzww + zzwzww + \dots$$

$$= \sum_{w \in \mathrm{im}\,\mathcal{D}} w$$

$$= (e_0^* \otimes 1)(I - S \otimes z - S^* \otimes w)^{-1}(e_0 \otimes 1)$$

---

1)

Theorem: This relationship holds for all Pythagorean languages.

NUMERICAL RADIUS

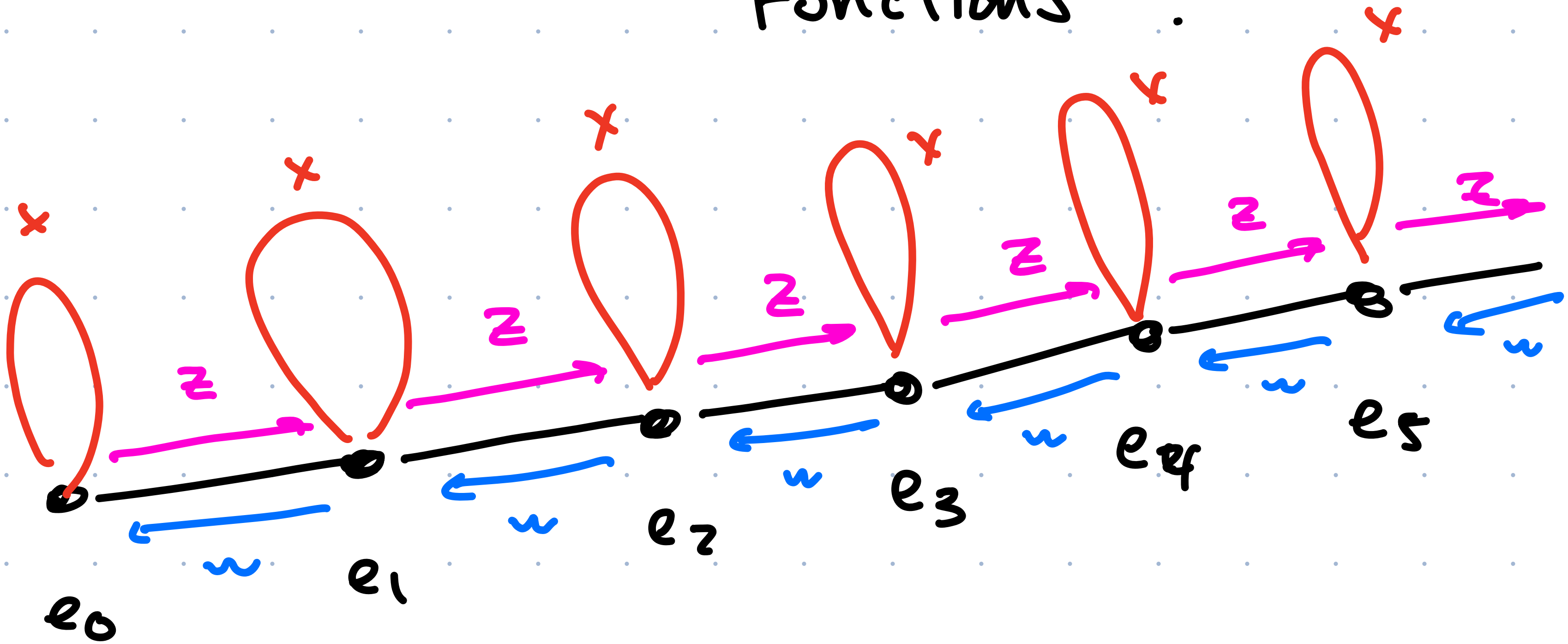
Let $p_n(z, z^*)$ denote the homogeneous polynomial of degree $2n$ in the irreducible Dyck words.

<u>Theorem:</u> The numerical radius of $Z \in M_n(\mathbb{C})$ is $\frac{1}{2} \limsup_{n \to \infty} \| p_n(Z, Z^*) \|^{1/2n}$

comment: really a corollary of a result of Ando. we showed $Y = Z(1 - Y)^{-1} Z^*$

$\Rightarrow Y = \sum_{w \in \mathrm{irr}\mathcal{D}} w$.

OTHER "GRAPHICAL FUNCTIONS"?



walks e_0 to e_0 are Motzkin paths.

$z^* = w, \quad x^* = x.$

Thank
You.